

DENSITY OF RATIONAL POINTS ON DEL PEZZO SURFACES OF DEGREE ONE

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ABSTRACT. We state conditions under which the set $S(k)$ of k -rational points on a del Pezzo surface S of degree 1 over an infinite field k of characteristic not equal to 2 or 3 is Zariski dense. For example, it suffices to require that the elliptic fibration $S \rightarrow \mathbb{P}^1$ induced by the anticanonical map has a nodal fiber over a k -rational point of \mathbb{P}^1 . It also suffices to require the existence of a point in $S(k)$ that does not lie on six exceptional curves of S and that has order 3 on its fiber of the elliptic fibration. This allows us to show that within a parameter space for del Pezzo surfaces of degree 1 over \mathbb{R} , the set of surfaces S defined over \mathbb{Q} for which the set $S(\mathbb{Q})$ is Zariski dense, is dense with respect to the real analytic topology. We also include conditions that may be satisfied for every del Pezzo surface S and that can be verified with a finite computation for any del Pezzo surface S that does satisfy them.

1. INTRODUCTION

A del Pezzo surface over a field k is a smooth, projective, geometrically integral surface S over k with ample anticanonical divisor $-K_S$; the degree of S is defined to be the self-intersection number $d = K_S^2 \geq 1$. A del Pezzo surface over a field k is minimal over k if and only if there is no birational morphism to a del Pezzo surface of higher degree over k . A del Pezzo surface of degree d is geometrically isomorphic to \mathbb{P}^2 blown up at $9 - d$ points in general position, or to $\mathbb{P}^1 \times \mathbb{P}^1$ if $d = 8$. Conversely, every smooth, projective surface that is geometrically birationally equivalent to \mathbb{P}^2 is birationally equivalent over the groundfield to a del Pezzo surface or a conic bundle [10].

Segre and Manin proved that every del Pezzo surface S of degree $d \geq 2$ over a field k with a k -rational point that is assumed to be in general position in the case $d = 2$, is unirational over k , that is, there is a dominant rational map $\mathbb{P}^2 \rightarrow S$ (see [27, 28] for $d = 3$ and $k = \mathbb{Q}$, see [18, Theorem 29.4 and 30.1] for $d = 2$ and $d \geq 5$, as well as $d = 3, 4$ under the assumption that k is large enough, and see [14, Theorem 1.1] and [26, Proposition 5.19] for $d = 3$ and $d = 4$ in general). On the other hand, even though del Pezzo surfaces of degree 1 always have a rational point, we do not know whether there exists a minimal del Pezzo surface S of degree 1, not birationally equivalent to a conic bundle, that is unirational over its ground field. If k is infinite, then unirationality of S implies that the set $S(k)$ of k -rational points on S is Zariski dense. The following conjecture states that this weaker property may hold for all del Pezzo surfaces of degree 1 over an infinite field; for number fields it follows from the conjecture by Colliot-Thélène and Sansuc that the Brauer–Manin obstruction to weak approximation is the only one for geometrically rational varieties [3, Conjecture d), p. 319], which may in fact hold more generally for geometrically rationally connected varieties over number fields [4, p. 3].

Conjecture 1.1. *If S is a del Pezzo surface of degree 1 over an infinite field k , then the set $S(k)$ is Zariski dense in S .*

The primary goal of this paper is to state conditions under which this conjecture holds.

Let k be a field of characteristic not equal to 2 or 3, and S a del Pezzo surface of degree 1 over k with a canonical divisor K_S . Then the linear system $|-3K_S|$ induces an embedding of S in the weighted projective space $\mathbb{P} = \mathbb{P}_k(2, 3, 1, 1)$ with coordinates x, y, z, w . More precisely, there are homogeneous polynomials $f, g \in k[z, w]$ of degrees 4 and 6, respectively, such that S is isomorphic to the smooth sextic in \mathbb{P} given by

$$(1) \quad y^2 = x^3 + f(z, w)x + g(z, w).$$

For some special families of del Pezzo surfaces of degree 1 it is known that the set of rational points is Zariski dense. Of course this includes surfaces that are not minimal and surfaces that are birationally equivalent to a conic bundle. Furthermore, A. Várilly-Alvarado proves in [38, Theorem 2.1] that if we have $k = \mathbb{Q}$ and $f = 0$ and g satisfies some technical conditions (for example, $g = az^6 + bw^6$ for nonzero integers $a, b \in \mathbb{Z}$ with $3ab$ not a square, or with a and b relatively prime and $9 \nmid ab$ [38, Theorem 1.1]), then the set of \mathbb{Q} -rational points on the surface S given by (1) is Zariski dense if one also assumes that Tate–Shafarevich groups of elliptic curves over \mathbb{Q} with j -invariant 0 are finite (cf. Example 7.3).

M. Ulas [34, 35], as well as M. Ulas and A. Togbé [36, Theorem 2.1], also give various conditions on the homogeneous polynomials $f, g \in \mathbb{Q}[z, w]$ for the set of rational points on the surface $S \subset \mathbb{P}(2, 3, 1, 1)$ over \mathbb{Q} given by (1) to be Zariski dense. Besides hypotheses that imply that S is not smooth or not minimal, all their conditions imply that either (i) $f = 0$ and $g(t, 1)$ is monic, or (ii) $g(t, 1)$ has degree at most 4, or (iii) $f = 0$ and g vanishes on a rational point of \mathbb{P}^1 . E. Jabara generalizes Ulas’ work on case (iii) in [12, Theorems C and D] and treats the general case over \mathbb{Q} with $g(t, 1)$ monic.

Our techniques in this paper are an independent generalization of a geometric interpretation of Ulas’ work on case (iii) (see Remark 2.7). The projection $\varphi: \mathbb{P}(2, 3, 1, 1) \rightarrow \mathbb{P}^1(z, w)$ is a morphism on the complement U of the line given by $z = w = 0$ in $\mathbb{P}(2, 3, 1, 1)$. For any point $Q \in S(k)$ not equal to $\mathcal{O} = [1 : 1 : 0 : 0]$, we let $\mathcal{C}_Q(5)$ denote the family of sections of $U \rightarrow \mathbb{P}^1$ that meet S at Q with multiplicity at least 5; we will see that $\mathcal{C}_Q(5)$ has the structure of an affine curve of genus at most 1 (see paragraph containing (8)).

The restriction $\varphi|_S: S \rightarrow \mathbb{P}^1$ corresponds to the linear system $|-K_S|$ and induces an elliptic fibration $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ of the blow-up \mathcal{E} of S at the unique base point \mathcal{O} of $\varphi|_S$. The exceptional curve on \mathcal{E} above \mathcal{O} is a section, also denoted by \mathcal{O} . For any $t = [z_0 : w_0] \in \mathbb{P}^1$, the fiber \mathcal{E}_t is isomorphic to the intersection of S with the plane H_t given by $w_0 z = z_0 w$; the set $\mathcal{E}_t^{\text{ns}}(k)$ of smooth k -points on \mathcal{E}_t naturally carries a group structure characterized by the property that three points in $H_t \cap S$ sum to the identity \mathcal{O} if and only if they are collinear. Our first main result is the following.

Theorem 1.2. *Let k be an infinite field of characteristic not equal to 2 or 3. Let $S \subset \mathbb{P}(2, 3, 1, 1)$ be a del Pezzo surface given by (1) with $f, g \in k[z, w]$, and $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ the elliptic fibration induced by the anticanonical map $S \rightarrow \mathbb{P}^1$. Let $Q \neq [1 : 1 : 0 : 0]$ be a point on S and $\mathcal{C}_Q(5)$ the curve of those sections of the projection $U \rightarrow \mathbb{P}^1$ that meet S at the point Q with multiplicity at least 5. Set $t = \pi(Q)$. Suppose that the following statements hold.*

- *The order of Q in $\mathcal{E}_t^{\text{ns}}(k)$ is at least 3.*
- *If the order of Q in $\mathcal{E}_t^{\text{ns}}(k)$ is at least 4, then $\mathcal{C}_Q(5)$ has infinitely many k -points.*
- *If the order of Q in $\mathcal{E}_t^{\text{ns}}(k)$ is 5, then the characteristic of k does not equal 5.*
- *If the order of Q in $\mathcal{E}_t^{\text{ns}}(k)$ is 3 or 5, then Q does not lie on six (-1) -curves of S .*

Then the set $S(k)$ of k -points on S is Zariski dense in S .

Note that all four assumptions of Theorem 1.2 are hypotheses on the point Q . Given S , we provide an explicit zero-dimensional scheme of which the points correspond to the (-1) -curves of S going through Q (cf. Remark 2.6), so the first and the last two conditions of Theorem 1.2 are easy to check. For a given del Pezzo surface $S \subset \mathbb{P}(2, 3, 1, 1)$ of degree 1 over a field k on which $S(k)$ is dense, the set of those points $Q \in S(k)$ that satisfy these three conditions is also dense; Theorem 1.2 provides a proof of Zariski density of $S(k)$ as soon as $\mathcal{C}_Q(5)(k)$ is infinite for one of these points Q . In light of Conjecture 1.1, it may be true that for *every* del Pezzo surface S of degree 1, there exists a point Q satisfying the conditions of Theorem 1.2, thus reducing the verification of Zariski density of $S(k)$ to a finite computation. Theorem 1.2 is the first result that states sufficient conditions for the set of rational points on a *general* del Pezzo surface of degree 1 to be Zariski dense.

Note that if the order of Q in $\mathcal{E}_0^{\text{ns}}(k)$ is 3 and Q does not lie on six (-1) -curves of S , then the assumptions in Theorem 1.2 are automatically satisfied without any further condition on $\mathcal{C}_Q(5)$. Besides verifying Zariski density of rational points on explicit surfaces, Theorem 1.2 also implies the following two results.

Theorem 1.3. *Let $f_0, \dots, f_4, g_0, \dots, g_6 \in \mathbb{Q}$ be such that the surface $S \in \mathbb{P}(2, 3, 1, 1)$ given by*

$$(2) \quad y^2 = x^3 + \left(\sum_{i=0}^4 f_i z^i w^{4-i} \right) x + \sum_{j=0}^6 g_j z^j w^{6-j} = 0$$

is smooth. Then for each $\ell \in \{0, \dots, 4\}$, $m \in \{0, \dots, 6\}$, and $\varepsilon > 0$, there exist $\lambda, \mu \in \mathbb{Q}$ with $|\lambda - f_\ell| < \varepsilon$ and $|\mu - g_m| < \varepsilon$ such that the surface $S' \in \mathbb{P}(2, 3, 1, 1)$ given by (2) with the two values f_ℓ and g_m replaced by λ and μ , respectively, is smooth and the set $S'(\mathbb{Q})$ is Zariski dense in S' .

Theorem 1.4. *Suppose k is an infinite field of characteristic not equal to 2 or 3. If S is a del Pezzo surface of degree 1 and the associated elliptic fibration $\mathcal{E} \rightarrow \mathbb{P}^1$ has a nodal fiber over a rational point in \mathbb{P}^1 , then $S(k)$ is Zariski dense in S .*

Our strategy to prove Theorem 1.2 is to exhibit a rational map $\sigma: \mathcal{C}_Q(5) \rightarrow S$ of which the image has a component whose strict transform on \mathcal{E} is a multisection of π of infinite order (cf. [1]). In the next section, we will construct σ . For any explicit surface S with a point Q , it is easy to check whether $\sigma(\mathcal{C}_Q(5))$ has a horizontal component, and if so, whether that component is a multisection of infinite order. Since this is indeed the case for some specific examples, we may already conclude that it is true for S and Q sufficiently general. To show that the image $\sigma(\mathcal{C}_Q(5))$ always has a horizontal component under the conditions in Theorem 1.2, we first choose a completion $\overline{\mathcal{C}}_Q(5)$ of the affine curve $\mathcal{C}_Q(5)$ and show that the added points correspond naturally to limits of the sections in $\mathcal{C}_Q(5)$, which allows us to show that σ extends to the extra points in $\overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$, sending them to $-4Q$ or $-5Q$ on the fiber of π containing Q (Section 3). This allows us to characterize all cases where no component of $\overline{\mathcal{C}}_Q(5)$ has a horizontal image under σ in Sections 4 and 5. In Section 6, we show that the base change of $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ by a curve of genus at most 1 has no nonzero torsion sections. Finally, we apply this to a horizontal component of $\sigma(\overline{\mathcal{C}}_Q(5))$ to prove all our main results in Section 7.

While we consider only surfaces given by (1) that are smooth, i.e., del Pezzo surfaces of degree 1, one could also consider *generalized del Pezzo surfaces* of degree 1, which have a birational model given by (1) that may have isolated rational double points. As for del Pezzo surfaces, there is a natural elliptic fibration on the blow-up of a generalized del Pezzo surface at the point corresponding to \mathcal{O} . All our results up to and including Section 5 also hold for generalized del Pezzo surfaces of degree 1, as long as we assume that the point Q does not lie on a reducible fiber, with the exception of Proposition 5.3; its proof shows that there is one more singular surface that we should add to the list of examples where $\sigma(\overline{\mathcal{C}}_Q(5))$ is not horizontal. One can actually generalize many of our results to the case that Q lies on a reducible fiber, but given the significant amount of additional computations required, this will be presented in a later paper.

In the proof of Theorem 1.4, we will view the family of the curves $\overline{\mathcal{C}}_Q(5)$ as Q runs through the points on the nodal fiber as an elliptic surface. We may also consider the family of *all* curves $\overline{\mathcal{C}}_Q(5)$ as an elliptic threefold over S , possibly adding some extra component in some fibers to achieve flatness. This threefold has real points for any surface S over \mathbb{R} ; it would be interesting to study the Hasse principle and weak approximation for this elliptic threefold.

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2. A FAMILY OF SECTIONS

By a variety over a field we will mean a separated scheme of finite type over that field. In particular, we do not assume that varieties are irreducible or reduced. As all varieties that we will encounter are connected, the word component will always mean irreducible component. Curves are varieties of which all components have dimension 1 and surfaces are varieties of which all components have dimension 2.

Let k be a field of characteristic not equal to 2 or 3 and let \mathbb{P} denote the weighted projective space $\mathbb{P}(2, 3, 1, 1)$ over k with coordinates x, y, z, w . The subset $Z \subset \mathbb{P}$ given by $z = w = 0$ contains the two singular points $[1 : 0 : 0 : 0]$ and $[0 : 1 : 0 : 0]$ of \mathbb{P} , so the complement $U = \mathbb{P} - Z$ is nonsingular. The projection $\varphi: \mathbb{P} \rightarrow \mathbb{P}^1(z, w)$ is well defined on U . Let \mathcal{C} denote the family of all curves C in U for which the restriction $\varphi|_C: C \rightarrow \mathbb{P}^1$ is an isomorphism, that is, \mathcal{C} is the family of sections of $\varphi: U \rightarrow \mathbb{P}^1$.

Lemma 2.1. *There is a bijection $\mathbb{A}^7(k) \rightarrow \mathcal{C}$ sending the point $(x_0, y_0, a, b, c, p, q)$ to the curve defined by*

$$(3) \quad x = qz^2 + pzw + x_0w^2 \quad \text{and} \quad y = cz^3 + bz^2w + azw^2 + y_0w^3.$$

Proof. Clearly, the described map is well defined and injective. To show surjectivity, let $\sigma: \mathbb{P}^1 \rightarrow U$ be a section of $\varphi: U \rightarrow \mathbb{P}^1$ containing Q . If we set $t = z/w$, then there are polynomials $r_1, r_2, s_1, s_2 \in k[t]$ such that σ is given on $\mathbb{A}^1(t) \subset \mathbb{P}^1$ by

$$t \mapsto \left[\frac{r_1(t)}{s_1(t)} : \frac{r_2(t)}{s_2(t)} : t : 1 \right].$$

The fact that the image C of σ is contained in U implies that s_1 and s_2 are constant and the degrees of r_1 and r_2 bounded by 2 and 3 respectively. This shows that indeed there are $x_0, y_0, a, b, c, p, q \in k$ such that C is given by (3). \square

Let $f, g \in k[z, w]$ be homogeneous of degree 4 and 6, respectively, and let $S \subset \mathbb{P}$ be the surface given by (1). The number of (-1) -curves on S is finite. Over a separable closure k^{sep} of k there are 240 such curves, characterized by the following lemma.

Lemma 2.2. *The curves in \mathcal{C} that are contained in S are exactly the (-1) -curves of S that are defined over k .*

Proof. The (-1) -curves are defined over a separable extension of k by [5, Theorem 1]. This shows that the assumption that k be perfect is not necessary in [37, Thm. 1.2], which therefore implies that the (-1) -curves on $S_{k^{\text{sep}}}$ are exactly the curves given by (3) for some $x_0, y_0, a, b, c, p, q \in k^{\text{sep}}$, which also follows from [30, Lemma 10.9]. The lemma follows from taking Galois invariants. \square

Proposition 2.3. *For each curve $C \in \mathcal{C}$ that is not contained in S , we have $C \cdot S = 6$.*

Proof. The equations (3) show that C has degree 6. Also, C is contained in U , so the intersection $C \cap S$ with the hypersurface S of degree 6 is contained in U , which is smooth. Therefore, intersection multiplicities are defined as usual, and the weighted analogue of Bézout's Theorem gives $C \cdot S = \mu^{-1}(\deg C) \cdot (\deg S)$, where $\mu = 6$ is the product of the weights of \mathbb{P} . The statement follows. \square

The intersection $S \cap Z$ consists of the single point $\mathcal{O} = [1 : 1 : 0 : 0]$. For any point $Q \in S(k) - \{\mathcal{O}\}$, and for $1 \leq n \leq 6$, we let $\mathcal{C}_Q(n)$ denote the subset of \mathcal{C} consisting of all curves C that intersect S at Q with multiplicity at least n . Note that for $n = 5$ this coincides with the definition of $\mathcal{C}_Q(5)$ in the introduction.

Let $Q \in S(k) - \{\mathcal{O}\}$. After applying an automorphism of \mathbb{P}^1 (and the corresponding automorphism of \mathbb{P}), we assume without loss of generality that $\varphi(Q) = 0 = [0 : 1]$, say $Q = [x_0 : y_0 : 0 : 1]$ for some $x_0, y_0 \in k$. Under the bijection $\mathcal{C} \cong \mathbb{A}^7$ of Lemma 2.1, the subset $\mathcal{C}_Q(1)$ corresponds with the subset of points of \mathbb{A}^7 of which the first two coordinates equal x_0 and y_0 , respectively, and this set projects isomorphically to $\mathbb{A}^5(a, b, c, p, q)$. From now on we will freely identify $\mathcal{C}_Q(1)$ with $\mathbb{A}^5(a, b, c, p, q)$ through the induced bijection.

As in the introduction, we let \mathcal{E} denote the blow-up of S at \mathcal{O} and $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ the elliptic fibration induced by the anticanonical map $\varphi|_S: S \rightarrow \mathbb{P}^1$. We will sometimes identify the fiber \mathcal{E}_t above $t = [z_0 : w_0]$ with its isomorphic image on S , equal to the intersection of S with the hyperplane H_t given by $w_0z = z_0w$ and denoted by S_t . The intersection $H_t \cap S$ is given by a Weierstrass equation; in particular, all fibers are irreducible, and therefore all singular fibers have

type I_1 or II . Whenever we speak of vertical or horizontal curves or of fibers on S or \mathcal{E} , we refer to this fibration. We write

$$\begin{aligned} f &= f_4 z^4 + f_3 z^3 w + \cdots + f_0 w^4, \\ g &= g_6 z^6 + g_5 z^5 w + \cdots + g_0 w^6, \end{aligned}$$

so the fiber \mathcal{E}_0 above $t = 0$, containing Q , is given by $y^2 = x^3 + f_0 x + g_0$.

We can give equations for $\mathcal{C}_Q(n)$ inside $\mathcal{C}_Q(1) \cong \mathbb{A}^5$ as follows. Note that $t = z/w$ is a local parameter at the point $[0 : 1]$ on \mathbb{P}^1 . Hence, around Q , the curve in $\mathcal{C}_Q(1)$ corresponding to (a, b, c, p, q) is parametrized by

$$(4) \quad \begin{cases} x &= qt^2 + pt + x_0, \\ y &= ct^3 + bt^2 + at + y_0, \\ z &= t, \\ w &= 1. \end{cases}$$

For $0 \leq i \leq 6$, let $F_i \in k[a, b, c, p, q]$ be the coefficient of t^i in

$$(5) \quad -y^2 + x^3 + f(t, 1)x + g(t, 1),$$

with x and y as in (4). Then we have

$$(6) \quad \begin{aligned} F_0 &= 0, \\ F_1 &= -2y_0 a + (3x_0^2 + f_0)p + f_1 x_0 + g_1, \\ F_2 &= -a^2 - 2y_0 b + 3x_0 p^2 + f_1 p + (3x_0^2 + f_0)q + f_2 x_0 + g_2, \\ F_3 &= -2ab - 2y_0 c + p^3 + 6x_0 p q + f_2 p + f_1 q + f_3 x_0 + g_3, \\ F_4 &= -2ac - b^2 + 3p^2 q + f_3 p + 3x_0 q^2 + f_2 q + f_4 x_0 + g_4, \\ F_5 &= -2bc + 3p q^2 + f_4 p + f_3 q + g_5, \\ F_6 &= -c^2 + q^3 + f_4 q + g_6, \end{aligned}$$

and the variety $\mathcal{C}_Q(n)$ is given by inside $\mathcal{C}_Q(1) = \mathbb{A}^5$ by $F_1 = F_2 = \cdots = F_{n-1} = 0$.

For every integer $j \geq 2$, let $\Phi_j \in k[x]$ be the factor of the j -th division polynomial of the fiber \mathcal{E}_0 that corresponds to the *primitive* j -torsion. In particular, the polynomials $\Phi_2, \Phi_3, \Phi_2\Phi_4, \Phi_5$, and $\Phi_2\Phi_3\Phi_6$ are the j -th division polynomials for $j = 2, 3, 4, 5, 6$, respectively. We have

$$\begin{aligned} \Phi_2 &= 4(x^3 + f_0 x + g_0), \\ \Phi_3 &= 3x^4 + 6f_0 x^2 + 12g_0 x - f_0^2, \\ \Phi_4 &= 2(x^6 + 5f_0 x^4 + 20g_0 x^3 - 5f_0^2 x^2 - 4f_0 g_0 x - f_0^3 - 8g_0^2), \\ \Phi_5 &= 5x^{12} + 62f_0 x^{10} + 380g_0 x^9 - 105f_0^2 x^8 + 240f_0 g_0 x^7 + (-300f_0^3 - 240g_0^2)x^6 \\ &\quad - 696f_0^2 g_0 x^5 + (-125f_0^4 - 1920f_0 g_0^2)x^4 + (-80f_0^3 g_0 - 1600g_0^3)x^3 \\ &\quad + (-50f_0^5 - 240f_0^2 g_0^2)x^2 + (-100f_0^4 g_0 - 640f_0 g_0^3)x + f_0^6 - 32f_0^3 g_0^2 - 256g_0^4, \\ \Phi_6 &= x^{12} + 22f_0 x^{10} + 220g_0 x^9 - 165f_0^2 x^8 - 528f_0 g_0 x^7 + (-92f_0^3 - 1776g_0^2)x^6 \\ &\quad + 264f_0^2 g_0 x^5 + (-185x^4 - 960f_0 g_0^2)x^4 + (-80f_0^3 g_0 - 320g_0^3)x^3 \\ &\quad + (-90f_0^5 - 624f_0^2 g_0^2)x^2 + (-132f_0^4 g_0 - 896f_0 g_0^3)x - 3f_0^6 - 96f_0^3 g_0^2 - 512g_0^4. \end{aligned}$$

Note that these polynomials are also defined if \mathcal{E}_0 is singular. For more compact notation, we set $\phi_j = \Phi_j(x_0)$ for all $j \geq 2$, as well as

$$\psi = \frac{1}{2}\Phi_2'(x_0), \quad h_i = (f_i x_0 + g_i)\phi_2^{i-1}, \quad l_i = f_i \phi_2^i - h_i \psi,$$

for $1 \leq i \leq 6$, where we use $f_5 = f_6 = 0$. We have

$$(7) \quad \begin{aligned} 4\phi_3 &= 12x\phi_2 - \psi^2, \\ \phi_4 &= \psi\phi_3 - \phi_2^2, \\ \phi_5 &= \phi_2^2\phi_4 - \phi_3^3, \\ \phi_6 &= \phi_5 - \phi_4^2. \end{aligned}$$

Lemma 2.4. *If $y_0 \neq 0$, then the projection of $\mathbb{A}^5(a, b, c, p, q)$ onto its last two coordinates induces an isomorphism $\mathcal{C}_Q(4) \rightarrow \mathbb{A}^2(p, q)$. The inverse is determined by*

$$\begin{aligned} a &= \frac{\psi p + 2h_1}{4y_0}, \\ b &= \frac{\psi\phi_2 q + 2\phi_3 p^2 + 2l_1 p + 2h_2 - 2h_1^2}{4y_0\phi_2}, \\ c &= \frac{\zeta q + \eta}{2y_0\phi_2^2}, \quad \text{with} \\ \zeta &= \phi_2(2\phi_3 p + l_1), \\ \eta &= -\phi_4 p^3 - (2h_1\phi_3 + l_1\psi)p^2 + (l_2 - 2h_1l_1 + h_1^2\psi)p + h_3 - 2h_1h_2 + 2h_1^3. \end{aligned}$$

Proof. Since F_1 is linear in a , the projection of $\mathcal{C}_Q(1) = \mathbb{A}^5$ onto $\mathbb{A}^4(b, c, p, q)$ induces an isomorphism $\mathcal{C}_Q(2) \rightarrow \mathbb{A}^4(b, c, p, q)$, of which the inverse is determined by the given expression for a . The image of $\mathcal{C}_Q(3)$ in $\mathbb{A}^4(b, c, p, q)$ has a defining equation that is linear in b , as F_2 is linear in b and F_1 is independent of b . Therefore, the projection of $\mathcal{C}_Q(2) \cong \mathbb{A}^4(b, c, p, q)$ onto $\mathbb{A}^3(c, p, q)$ induces an isomorphism $\mathcal{C}_Q(3) \rightarrow \mathbb{A}^3(c, p, q)$, of which the inverse is determined by the given expressions for a and b . Finally, the defining equation of the image of $\mathcal{C}_Q(4)$ in $\mathbb{A}^3(c, p, q)$ is linear in c , as F_3 is linear in c and F_1 and F_2 are independent of c . Therefore, the projection of $\mathcal{C}_Q(3) \cong \mathbb{A}^3(c, p, q)$ onto $\mathbb{A}^2(p, q)$ induces an isomorphism $\mathcal{C}_Q(4) \rightarrow \mathbb{A}^2(p, q)$ with inverse as claimed. \square

From now on we will assume $y_0 \neq 0$, or equivalently $\phi_2 \neq 0$, and we identify $\mathcal{C}_Q(4)$ with $\mathbb{A}^2(p, q)$ through the isomorphism of Lemma 2.4. We may eliminate the variables a, b, c from the equation $F_4 = 0$; after multiplying all coefficients by ϕ_2^3 , we find that the variety $\mathcal{C}_Q(5) \subset \mathcal{C}_Q(4) \cong \mathbb{A}^2(p, q)$ is defined by

$$(8) \quad c_1 q^2 + (c_2 p^2 + c_3 p + c_4) q = c_5 p^4 + c_6 p^3 + c_7 p^2 + c_8 p + c_9$$

with

$$\begin{aligned} c_1 &= \phi_2^2 \phi_3, \\ c_2 &= -3\phi_2 \phi_4, \\ c_3 &= -2\phi_2(l_1\psi + 2h_1\phi_3), \\ c_4 &= \phi_2(h_1^2\psi - 2l_1h_1 + l_2), \\ c_5 &= \phi_3^2 - \phi_4\psi, \\ c_6 &= 2l_1\phi_3 - 2h_1\phi_2^2 - 4h_1\phi_4 - l_1\psi^2, \\ c_7 &= h_1^2\psi^2 - 2(3h_1^2 - h_2)\phi_3 - (4l_1h_1 - l_2)\psi + l_1^2, \\ c_8 &= (4h_1^3 - 2h_1h_2)\psi - 6l_1h_1^2 + 2l_1h_2 + 2l_2h_1 - l_3, \\ c_9 &= 5h_1^4 - 6h_1^2h_2 + 2h_1h_3 + h_2^2 - h_4. \end{aligned}$$

As we assumed $y_0, \phi_2 \neq 0$ and the characteristic of k is not 2 or 3, the vanishing of ϕ_3 and ϕ_4 would imply that Q has both order 3 and 4 in $\mathcal{E}_0^{\text{ns}}(k)$, which is a contradiction, so the coefficients c_1 and c_2 do not both vanish, and $\mathcal{C}_Q(5)$ is a curve, though not necessarily reduced or irreducible. We will identify $\mathcal{C}_Q(5)$ with its image in $\mathbb{A}^2(p, q)$ and we view the coordinates a, b , and c as functions on $\mathcal{C}_Q(5)$, as given in Lemma 2.4.

Remark 2.5. The functions F_4, F_5 , and F_6 are regular on $\mathcal{C}_Q(4) \cong \mathbb{A}^2(p, q)$ and can therefore be identified with polynomials in $k[p, q]$.

Remark 2.6. The (-1) -curves on S going through Q correspond to the points of the scheme in $\mathcal{C}_Q(4) \cong \mathbb{A}^2(p, q)$ given by $F_4 = F_5 = F_6 = 0$.

Remark 2.7. A special case of Theorem 1.2 is Theorem 2.1(2) of [35]; indeed, when $f = 0$ and g vanishes at $[1 : 0] \in \mathbb{P}^1$, and $Q = [1 : 1 : 1 : 0]$, then the curve $\mathcal{C}_Q(5)$ is isomorphic to the curve given in Theorem 2.1(2) of [35]. The generalizations of this theorem given in [12, Theorems C and D] are also a special case of our Theorem 1.2, where one uses $Q = [0 : 1 : 1 : 0]$, which has order 3 in its fiber in the case of Theorem C. The proofs of Theorems C and D in [12] are incomplete, but they do work for surfaces S that are sufficiently general.

Every $C \in \mathcal{C}_Q(5)$ that is not contained in S , intersects S with multiplicity at least 5 at Q , so by Proposition 2.3, there is a unique sixth point of intersection, which is also defined over k . We define a rational map

$$\sigma: \mathcal{C}_Q(5) \rightarrow S$$

by sending $C \in \mathcal{C}_Q(5)$ to the sixth intersection point of C with S . The map σ is defined over k . By Proposition 2.3, it is well defined at each curve $C \in \mathcal{C}_Q(5)$ that is not contained in S , and thus not defined at at most 240 curves $C \in \mathcal{C}_Q(5)$. Every horizontal component of the image of σ , or its strict transform on \mathcal{E} , yields a multisection of the elliptic fibration $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$.

We can describe the map σ very explicitly. The curve $C \in \mathcal{C}_Q(5)$ corresponding to (a, b, c, p, q) is parametrized by (4). When we substitute the expressions of (4) into equation (5), we obtain $t^5(F_5 + F_6 t)$, so the sixth intersection point of $C \cap S$ corresponds to $t = -F_5/F_6$.

3. A COMPLETION OF THE FAMILY OF SECTIONS

We continue the notation of the previous section. In particular, the field k , the surface S , and the point Q are as before, and so are the objects that depend on them, including the elliptic fibration $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$, the elements $\psi, \phi_j, c_i \in k$, the curve $\mathcal{C}_Q(5) \subset \mathbb{A}^2(p, q)$, the functions a, b, c, F_i on $\mathcal{C}_Q(5)$, and the map $\sigma: \mathcal{C}_Q(5) \rightarrow S$.

Note that the union

$$(9) \quad \bigcup_{C \in \mathcal{C}_Q(5)} C$$

in \mathbb{P} is the image of the morphism $\gamma: \mathcal{C}_Q(5) \times \mathbb{P}^1 \rightarrow \mathbb{P}$ given by

$$((p, q), [z : w]) \mapsto [x : y : z : w],$$

with x and y as in (3). We will define a projective completion of $\mathcal{C}_Q(5)$ and extend γ to a conic bundle over this completion. Let $\bar{p}, \bar{q}, \bar{r}$ be the coordinates of the weighted projective space $\mathbb{P}(1, 2, 1)$, and let $\mathbb{H} \rightarrow \mathbb{P}(1, 2, 1)$ be the blow-up at the singular point $[0 : 1 : 0]$. Since $\mathbb{P}(1, 2, 1)$ is isomorphic to a cone in \mathbb{P}^3 , the surface \mathbb{H} is smooth; it is in fact a Hirzebruch surface. By sending (p, q) to $[p : q : 1]$, we identify \mathbb{A}^2 with an open subset of $\mathbb{P}(1, 2, 1)$ and hence with an open subset of \mathbb{H} .

Let $\bar{\mathcal{C}}_Q(5)$ denote the completion of $\mathcal{C}_Q(5)$ inside \mathbb{H} . Note that the completion of $\mathcal{C}_Q(5)$ inside $\mathbb{P}(1, 2, 1)$ contains the singular point $[0 : 1 : 0]$ if and only if $c_1 = 0$, i.e., if and only if Q has order 3 in $\mathcal{E}_0^{\text{ns}}(k)$. Hence, if Q does not have order 3, we may identify $\bar{\mathcal{C}}_Q(5)$ with the completion of $\mathcal{C}_Q(5)$ inside $\mathbb{P}(1, 2, 1)$; as c_1, c_2 , and c_5 do not all vanish, this completion is given by

$$(10) \quad c_1 \bar{q}^2 + (c_2 \bar{p}^2 + c_3 \bar{p} \bar{r} + c_4 \bar{r}^2) \bar{q} = c_5 \bar{p}^4 + c_6 \bar{p}^3 \bar{r} + c_7 \bar{p}^2 \bar{r}^2 + c_8 \bar{p} \bar{r}^3 + c_9 \bar{r}^4.$$

We will now describe $\bar{\mathcal{C}}_Q(5)$ explicitly, also in the case that Q does have order 3.

Definition 3.1. Besides the functions $p = \bar{p}/\bar{r}$ and $q = \bar{q}/\bar{r}^2$ on $\bar{\mathcal{C}}_Q(5)$, and a, b, c as in Lemma 2.4, we define the functions

$$\begin{aligned} r &= p^{-1}, & a' &= ra, & a'' &= a, & a''' &= ra'', \\ q' &= qp^{-2}, & b' &= r^2b, & b'' &= sb, & b''' &= b'', \\ s &= q^{-1}, & c' &= r^3c, & c'' &= sc, & c''' &= rc'', \\ s' &= q'^{-1}, \end{aligned}$$

on $\bar{\mathcal{C}}_Q(5)$.

We let $\mathcal{C}'_Q(5)$, $\mathcal{C}''_Q(5)$, and $\mathcal{C}'''_Q(5)$ denote the affine open subsets of $\bar{\mathcal{C}}_Q(5)$ where the pairs (q', r) , (p, s) , and (r, s') are regular, respectively. Together with $\mathcal{C}_Q(5)$, they cover $\bar{\mathcal{C}}_Q(5)$, each corresponding to one of four standard affine parts of \mathbb{H} . Note that $r = \bar{r}/\bar{p}$ and $q' = \bar{q}/\bar{p}^2$ on $\bar{\mathcal{C}}_Q(5)$, so $\mathcal{C}'_Q(5)$ is the affine part of $\bar{\mathcal{C}}_Q(5)$ corresponding to $\bar{p} \neq 0$, given by

$$c_1q'^2 + (c_2 + c_3r + c_4r^2)q' = c_5 + c_6r + c_7r^2 + c_8r^3 + c_9r^4$$

in $\mathbb{A}^2(q', r)$. If Q does not have order 3 in $\mathcal{E}_0^{\text{ns}}(k)$, so that $\phi_3 \neq 0$, then we have $\bar{\mathcal{C}}_Q(5) = \mathcal{C}_Q(5) \cup \mathcal{C}'_Q(5)$. If Q does have order 3, then $\phi_3 = 0$, and the subsets $\mathcal{C}''_Q(5) \subset \mathbb{A}^2(p, s)$ and $\mathcal{C}'''_Q(5) \subset \mathbb{A}^2(r, s')$ are given by

$$c_2p^2 + c_3p + c_4 = s(c_5p^4 + c_6p^3 + c_7p^2 + c_8p + c_9),$$

and

$$c_2 + c_3r + c_4r^2 = s'(c_5 + c_6r + c_7r^2 + c_8r^3 + c_9r^4),$$

respectively. It follows from Lemma 2.4 that the triples (a, b, c) , (a', b', c') , (a'', b'', c'') , and (a''', b''', c''') are regular on $\mathcal{C}_Q(5)$, $\mathcal{C}'_Q(5)$, $\mathcal{C}''_Q(5)$, and $\mathcal{C}'''_Q(5)$, respectively.

We let Γ be the conic bundle over $\bar{\mathcal{C}}_Q(5)$ that is the glueing of the trivial \mathbb{P}^1 -bundles $\mathcal{C}_Q(5) \times \mathbb{P}^1(z, w)$ and $\mathcal{C}'_Q(5) \times \mathbb{P}^1(z', w')$, the conic bundle Γ° over $\mathcal{C}''_Q(5)$ given inside $\mathcal{C}''_Q(5) \times \mathbb{P}^2(u_0, u_1, u_2)$ by $s(C)u_0u_2 = u_1^2$, and the conic bundle Γ'° over $\mathcal{C}'''_Q(5)$ given inside $\mathcal{C}'''_Q(5) \times \mathbb{P}^2(u'_0, u'_1, u'_2)$ by $s'(C)u'_0u'_2 = u'^2_1$; the glueing is done according to the isomorphism

$$\begin{aligned} (\mathcal{C}_Q(5) \cap \mathcal{C}'_Q(5)) \times \mathbb{P}^1(z, w) &\rightarrow (\mathcal{C}_Q(5) \cap \mathcal{C}'_Q(5)) \times \mathbb{P}^1(z', w') \\ (C, [z : w]) &\mapsto (C, [p(C) \cdot z : w]) \end{aligned}$$

above $\mathcal{C}_Q(5) \cap \mathcal{C}'_Q(5)$, the isomorphism induced by the morphism

$$\begin{aligned} (\mathcal{C}_Q(5) \cap \mathcal{C}''_Q(5)) \times \mathbb{P}^1(z, w) &\rightarrow (\mathcal{C}_Q(5) \cap \mathcal{C}''_Q(5)) \times \mathbb{P}^2(u_0, u_1, u_2) \\ (C, [z : w]) &\mapsto (C, [q(C) \cdot z^2 : zw : w^2]) \end{aligned}$$

above $\mathcal{C}_Q(5) \cap \mathcal{C}''_Q(5)$, the isomorphism induced by the morphism

$$\begin{aligned} (\mathcal{C}''_Q(5) \cap \mathcal{C}'''_Q(5)) \times \mathbb{P}^2(u_0, u_1, u_2) &\rightarrow (\mathcal{C}''_Q(5) \cap \mathcal{C}'''_Q(5)) \times \mathbb{P}^2(u'_0, u'_1, u'_2) \\ (C, [u_0 : u_1 : u_2]) &\mapsto (C, [u_0 : p(C) \cdot u_1 : u_2]) \end{aligned}$$

and the induced isomorphisms above the other three intersections, which are easily checked to be isomorphisms indeed. We let τ denote the fibration $\Gamma \rightarrow \bar{\mathcal{C}}_Q(5)$. The conic bundle Γ was constructed from the trivial bundle $\bar{\mathcal{C}}_Q(5) \times \mathbb{P}^1$ by a sequence of blow-ups and blow-downs, chosen such that the map γ extends to a morphism $\Gamma \rightarrow \mathbb{P}$. The following proposition makes this explicit.

Proposition 3.2. *The map γ extends to a morphism $\Gamma \rightarrow \mathbb{P}$ that is given on $\mathcal{C}'_Q(5) \times \mathbb{P}^1(z', w')$ by sending $(C, [z', w'])$ to*

$$[q'(C)z'^2 + z'w' + x_0w'^2 : c'(C)z'^3 + b'(C)z'^2w' + a'(C)z'w'^2 + y_0w'^3 : r(C)z' : w'],$$

on $\Gamma^\circ \subset \mathcal{C}''_Q(5) \times \mathbb{P}^2(u_0, u_1, u_2)$ by sending $(C, [u_0 : u_1 : u_2])$ to

$$[u_2(u_0 + p(C)u_1 + x_0u_2) : u_2(c''(C)u_0u_1 + b''(C)u_0u_2 + a''(C)u_1u_2 + y_0u_2^2) : u_1 : u_2],$$

and on $\Gamma^{\circ'} \subset \mathcal{C}_Q'''(5) \times \mathbb{P}^2(u'_0, u'_1, u'_2)$ by sending $(C, [u'_0 : u'_1 : u'_2])$ to

$$[u'_2(u'_0 + u'_1 + x_0 u'_2) : u'_2(c'''(C)u'_0 u'_1 + b'''(C)u'_0 u'_2 + a'''(C)u'_1 u'_2 + y_0 u'^2_2) : r(C)u'_1 : u'_2].$$

Proof. It is easy to check that the given maps coincide with γ wherever they are well defined. Hence, it suffices to show that they are well defined on the claimed subsets.

Suppose the first map is not well defined at a point $((q', r), [z', w']) \in \mathcal{C}'_Q(5) \times \mathbb{P}^1(z', w')$. Then we have $w' = 0$, so $z' \neq 0$, and thus $r = c' = q = 0$. From Lemma 2.4 and $r = q' = 0$, we obtain $0 = c' = -\phi_4/(2y_0\phi_2^2)$, so we get $\phi_4 = 0$. From $r = q' = 0$, we find $c_5 = 0$, so we also have $\phi_3^2 = \phi_4\psi = 0$. This contradicts the fact that Q can not have both order 3 and 4 on $\mathcal{E}_0^{\text{ns}}(k)$, so the first map is well defined above $\mathcal{C}'_Q(5)$.

It is clear that the second map is well defined at any point $((p, s), [u_0 : u_1 : u_2]) \in \Gamma^\circ \subset \mathcal{C}''_Q(5) \times \mathbb{P}^2$ with $u_1 \neq 0$ or $u_2 \neq 0$. To see that it is also well defined at points with $[u_0 : u_1 : u_2] = [1 : 0 : 0]$, we identify \mathbb{P} with its image under the closed immersion to \mathbb{P}^{22} corresponding to $\mathcal{O}(6)$ on \mathbb{P} . Substituting the expressions for the second map into the 23 monomials of weighted degree 6 in the variables x, y, z , and w gives 23 polynomials of total degree 6 in u_0, u_1, u_2 , which after replacing u_1^2 by su_0u_2 are all divisible by u_2^3 . The composition $\Gamma^\circ \rightarrow \mathbb{P} \rightarrow \mathbb{P}^{22}$ is given by these 23 polynomials, each divided by u_2^3 . The coordinate corresponding to the monomial x^3 is given by $(u_0 + pu_1 + x_0u_2)^3$, which does not vanish, so this composition, and thus the map $\Gamma^\circ \rightarrow \mathbb{P}$, is well defined.

The third map is well defined whenever $u'_2 \neq 0$. On the other hand, if $u'_2 = 0$, then also $u'_1 = 0$, and one uses the composition $\Gamma^{\circ'} \rightarrow \mathbb{P} \rightarrow \mathbb{P}^{22}$ to check that the map $\Gamma^{\circ'} \rightarrow \mathbb{P}$ is well defined at points with $[u'_0 : u'_1 : u'_2] = [1 : 0 : 0]$, as in the previous. \square

By abuse of notation, we will denote the morphism $\Gamma \rightarrow \mathbb{P}$ of Proposition 3.2 by γ as well. Let T denote its image. Then T is the closure in \mathbb{P} of the image of $\mathcal{C}_Q(5) \times \mathbb{P}^1$, i.e., of the union (9). More precisely, set $\Omega = \overline{\mathcal{C}_Q(5)} - \mathcal{C}_Q(5)$. Then T equals the union of this union and the image under γ of the fibers of $\tau: \Gamma \rightarrow \overline{\mathcal{C}_Q(5)}$ over points on Ω ; these images are described in Lemmas 3.3, 3.4, 3.5, and 3.6.

Lemma 3.3. *Each point $C \in \mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$ corresponds to $(q', r) = (\alpha, 0)$ for some root α of the polynomial $c_1q'^2 + c_2q' - c_5$; the map γ sends $(C, [z' : w']) \in \Gamma$ to $[x : y : z : w]$ with*

$$\begin{cases} x &= \alpha z'^2 + z'w' + x_0w'^2, \\ y &= y_0 \left(\frac{4\alpha\phi_2\phi_3 - 2\phi_4}{\phi_2^3} z'^3 + \frac{\alpha\psi\phi_2 + 2\phi_3}{\phi_2^2} z'^2w' + \frac{\psi}{\phi_2} z'w'^2 + w'^3 \right), \\ z &= 0, \\ w &= w', \end{cases}$$

and the image of the fiber $\tau^{-1}(C) \subset \Gamma$ under γ is a curve on T of degree 6 that intersects S at Q with multiplicity at least 5.

Proof. Let $C \in \mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$. The first part of the statement is obvious. From Lemma 2.4 we deduce

$$a' = a'(C) = \frac{\psi}{4y_0}, \quad b' = b'(C) = \frac{\psi\phi_2\alpha + 2\phi_3}{4y_0\phi_2}, \quad \text{and} \quad c' = c'(C) = \frac{2\phi_2\phi_3\alpha - \phi_4}{2y_0\phi_2^2}.$$

According to Proposition 3.2, the map γ sends $(C, [z' : w']) \in \mathcal{C}'_Q(5) \times \mathbb{P}^1(z', w')$ to $[x : y : 0 : 1]$ with $x = \alpha z'^2 + z'w' + x_0w'^2$ and $y = c'z'^3 + b'z'^2w' + a'z'w'^2 + y_0w'^3$. Using $4y_0^2 = \phi_2$, it is easy to check that the latter equals the expression given for y in the lemma.

The fact that the curve $D = \gamma(\tau^{-1}(C))$ in T has degree 6 follows from the fact that it lies inside the hyperplane given by $z = 0$, which is isomorphic to the weighted projective space $\mathbb{P}(2, 3, 1)$, and the intersection of D with a curve D' in this hyperplane given by $y = \lambda xw + \mu w^3$ yields three intersection points for general λ and μ , while Bézout's Theorem tells us that the product of the weights times the intersection number equals $(\deg D)(\deg D') = 18$.

Since the degree of D is 6, it is a full limit of images under γ of fibers of Γ , all of which intersect S at Q with multiplicity at least 5, so D does this as well. This can also be checked

computationally by substituting the parametrization given in the lemma into the polynomial

$$-y^2 + x^3 + f(z, w)x + g(z, w),$$

and checking that the coefficients of $z'^i w'^{6-i}$ vanish for $0 \leq i \leq 4$. \square

Recall that S_0 is the image of \mathcal{E}_0 on S , which is the intersection of S with the plane given by $z = 0$. The following two lemmas give more information about the image under γ of the fibers of τ above points in $\mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$ in the case that S_0 is singular. In particular, they show that S_0 is contained in T in this case.

Lemma 3.4. *Suppose \mathcal{E}_0 has a node. Then $\mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$ contains the point $C_1 = (\alpha_1, 0) \in \mathbb{A}^2(q', r)$ with*

$$\alpha_1 = \frac{f_0}{4(f_0 x_0 - 3g_0)}.$$

The map γ restricts to a birational morphism from the fiber $\tau^{-1}(C_1)$ to S_0 . If $\phi_3 = 0$, then C_1 is the only point in $\mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$. If $\phi_3 \neq 0$, then $\mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$ contains a unique second point $C_2 = (\alpha_2, 0) \in \mathbb{A}^2(q', r)$ with

$$\alpha_2 = \frac{f_0(2f_0 x_0 - 21g_0)}{4(f_0 x_0 - 3g_0)(2f_0 x_0 - 9g_0)};$$

the image under γ of the fiber $\tau^{-1}(C_2)$ is not contained in S .

Proof. Since \mathcal{E}_0 has a node, we have $4f_0^3 + 27g_0^2 = 0$ with $f_0, g_0 \neq 0$, so for $d = -\frac{3g_0}{2f_0}$ we have $f_0 = -3d^2$ and $g_0 = 2d^3$. The curve $\mathcal{E}_0 \cong S_0$ is given by $y^2 = (x - d)^2(x + 2d)$, and we have

$$\Phi_2 = 4(x - d)^2(x + 2d),$$

$$\Phi_3 = 3(x - d)^3(x + 3d),$$

$$\Phi_4 = 2(x - d)^5(x + 5d),$$

$$\Phi_5 = (x - d)^{10}(5x^2 + 50dx + 89d^2).$$

If $\phi_3 \neq 0$, then $c_1 \neq 0$ and the polynomial $c_1 q'^2 + c_2 q' - c_5$ factors as $c_1(q' - \alpha_1)(q' - \alpha_2)$ with $\alpha_1 = \frac{1}{4}(x_0 + 2d)^{-1}$ and $\alpha_2 = \frac{1}{4}(x_0 + 7d)(x_0 + 2d)^{-1}(x_0 + 3d)^{-1}$, which equal the expressions given in the proposition. If $\phi_3 = 0$, then $c_1 = 0$ and $x_0 = -3d$, so the only root of $c_1 q'^2 + c_2 q' - c_5$ is $\alpha_1 = c_5/c_2 = -\frac{1}{4}d^{-1}$, which equals the expression for α_1 given in the proposition. This proves that the points in $\mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$ are as claimed.

It follows from Lemma 3.3 and the identities above that the restriction of γ to $\{C_1\} \times \mathbb{P}^1(z', w')$ factors as the composition of the isomorphism

$$\{C_1\} \times \mathbb{P}^1(z', w') \rightarrow \mathbb{P}^1, \quad (C_1, [z' : w']) \mapsto [(x_0 - d)(z' + 2(x_0 + 2d)w') : 2y_0 w']$$

and the birational morphism

$$\mathbb{P}^1 \rightarrow S_0, \quad [s : 1] \mapsto [s^2 - 2d : s^3 - 3ds : 0 : 1].$$

This proves the second statement.

For the last statement, we assume $\phi_3 \neq 0$, take $\alpha = \alpha_2$ and substitute the corresponding parametrization of Lemma 3.3 in the equation

$$-y^2 + x^3 + f(z, w)x + g(z, w) = 0,$$

which defines S . The obtained equation in z' and w' , multiplied by

$$-16d^{-3}(x_0 - d)^{10}(x_0 + 2d)^5(x_0 + 3d)^3,$$

is

$$z'^5(\phi_5 \cdot z' + (x_0 - d)^6 \phi_2 \phi_3 \cdot w') = 0.$$

As the left-hand side does not vanish identically, the curve $\gamma(\tau^{-1}(C_2))$ is not contained in S . \square

Lemma 3.5. *Suppose \mathcal{E}_0 has a cusp. Then $\overline{\mathcal{C}}_Q(5)$ equals $\mathcal{C}_Q(5) \cup \mathcal{C}'_Q(5)$ and $\mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$ contains exactly one point, namely $C = (\frac{1}{4}x_0^{-1}, 0) \in \mathbb{A}^2(q', r)$. The map γ restricts to a birational morphism from the fiber of Γ above C to S_0 .*

Proof. Since \mathcal{E}_0 , or equivalently S_0 , has a cusp, we have $f_0 = g_0 = 0$. The cusp $[0 : 0 : 0 : 1]$ is the only point on S_0 with x -coordinate 0, so from $y_0 \neq 0$ we conclude $x_0 \neq 0$. From $c_1 = 48x_0^{10} \neq 0$ we conclude $\overline{\mathcal{C}}_Q(5) = \mathcal{C}_Q(5) \cup \mathcal{C}'_Q(5)$. The polynomial $c_1q'^2 + c_2q' - c_5$ factors as $3x_0^8(4x_0q' - 1)^2$ with the unique root $\alpha = (4x_0)^{-1}$, which implies that $C = (\frac{1}{4}x_0^{-1}, 0)$ is the only point in $\mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$. It follows from Lemma 3.3 that the restriction of γ to $\{C\} \times \mathbb{P}^1(z', w')$ factors as the composition of the isomorphism

$$\{C\} \times \mathbb{P}^1(z', w') \rightarrow \mathbb{P}^1, \quad (C, [z' : w']) \mapsto [z' + 2x_0w' : 2x_0w']$$

and the birational morphism $\mathbb{P}^1 \rightarrow S_0$ that sends $[s : 1]$ to $[x_0s^2 : y_0s^3 : 0 : 1]$. This proves the second statement. \square

The points of $\overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$ that are not handled by the previous lemmas are the points in $\overline{\mathcal{C}}_Q(5) - (\mathcal{C}'_Q(5) \cup \mathcal{C}_Q(5))$, that is, the points above the singular point $[0 : 1 : 0]$ in $\mathbb{P}(1, 2, 1)$. The next lemma takes care of these points.

Lemma 3.6. *For each point $C \in \overline{\mathcal{C}}_Q(5) - (\mathcal{C}'_Q(5) \cup \mathcal{C}_Q(5))$, the map γ sends the fiber $\tau^{-1}(C)$ to the curve in \mathbb{P} given by $z = 0$ and $4y_0y = \psi xw + (\phi_2 - \psi x_0)w^3$; this curve intersects S with multiplicity 3 at Q and nowhere else.*

Proof. If C lies in $\mathcal{C}''_Q(5) - (\mathcal{C}'_Q(5) \cup \mathcal{C}_Q(5))$, then it corresponds to a point $(p, 0) \in \mathbb{A}(p, s)$ and the fiber $\tau^{-1}(C)$ is given by $u_1^2 = 0$ in $\mathbb{P}^2(u_0, u_1, u_2)$. From Lemma 2.4 we find $4y_0b''(C) = \psi$, and γ sends $(C, [u : 0 : 1])$ to $[u + x_0 : b''(C)u + y_0 : 0 : 1]$ by Lemma 3.2. It follows that the image of the fiber is the claimed curve. In the affine plane given by $z = 0$ and $w \neq 0$, this curve is exactly the tangent line to the curve S_0 at Q . Note that the existence of C implies that Q has order 3 on $S_0^{\text{ns}}(k)$, so this tangent line intersects S_0 with multiplicity 3 at Q and nowhere else. As the curve intersects the surface S only in the curve S_0 , the lemma follows. If C lies in $\mathcal{C}'''_Q(5) - (\mathcal{C}'_Q(5) \cup \mathcal{C}_Q(5))$, then the argument is analogous. \square

Remark 3.7. Scheme theoretically, the image of the fiber of Γ above C in Lemma 3.6 is not reduced, given by $z^2 = 0$ and $4y_0y = \psi xw + (\phi_2 - \psi x_0)w^3$. This curve is also a limit of the images of other fibers and intersects S with multiplicity 6 at Q .

Remark 3.8. The map $\gamma : \mathcal{C}_Q(5) \times \mathbb{P}^1 \rightarrow \mathbb{P}$ extends naturally to a map $\mathcal{C}_Q(n) \times \mathbb{P}^1 \rightarrow \mathbb{P}$ for any $1 \leq n \leq 6$. By Lemma 2.4, the variety $\mathcal{C}_Q(4)$ is isomorphic to $\mathbb{A}^2(p, q)$. The computations we have done also show that the map $\mathcal{C}_Q(4) \times \mathbb{P}^1 \rightarrow \mathbb{P}$ naturally extends to a morphism from a conic bundle Δ over \mathbb{H} to \mathbb{P} , which restricts to the map $\Gamma \rightarrow \mathbb{P}$.

The rational map $\sigma : \mathcal{C}_Q(5) \dashrightarrow S$ from the end of Section 2 factors as $\sigma = \gamma \circ \rho$, where $\rho : \mathcal{C}_Q(5) \dashrightarrow \mathcal{C}_Q(5) \times \mathbb{P}^1(z, w)$ is a rational section of $\tau : \Gamma \rightarrow \overline{\mathcal{C}}_Q(5)$ that sends $C \in \mathcal{C}_Q(5)$ to $(C, [-F_5(C) : F_6(C)])$, where, for $0 \leq i \leq 6$, we view F_i as in (6) as a function on $\overline{\mathcal{C}}_Q(5)$. We will use this in Lemma 3.9 to show that σ extends to a map that is well defined at every point in $\overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$.

$$\begin{array}{ccccccc}
 \overline{\mathcal{C}}_Q(5) & \xleftarrow{\tau} & \Gamma & \xrightarrow{\gamma} & \mathbb{P} & \xleftarrow{\quad} & S \\
 \uparrow & \nearrow \rho & \uparrow & & \uparrow & & \uparrow \\
 \mathcal{C}_Q(5) & \xleftarrow{\quad} & \mathcal{C}_Q(5) \times \mathbb{P}^1 & \xrightarrow{\gamma} & U & \xleftarrow{\quad} & S - \{\mathcal{O}\} \\
 & \searrow \sigma & & & & &
 \end{array}$$

Proposition 3.9. *The rational map σ extends to a rational map $\overline{\mathcal{C}}_Q(5) \dashrightarrow S$ that is well defined at the points in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$. If $C \in \Omega$, then $\sigma(C) = -4Q \in S_0^{\text{ns}}(k) \subset S$ if S_0 has a cusp or S_0 has a node with $C = C_1$ as in Lemma 3.4, and $\sigma(C) = -5Q \in S_0^{\text{ns}}(k) \subset S$ otherwise.*

Proof. Let $C \in \Omega$. Then by Lemmas 3.3, 3.4, 3.5, 3.6, and Remark 3.7, the point C corresponds to the curve $\tau^{-1}(C)$ of degree 6 in the plane given by $z = 0$ in \mathbb{P} , with a parametrization given in Lemma 3.3 if $C \in \mathcal{C}'_Q(5) - \mathcal{C}_Q(5)$, and nonreduced if $C \in \overline{\mathcal{C}}_Q(5) - (\mathcal{C}_Q(5) \cup \mathcal{C}'_Q(5))$. We will denote this curve by C as well, motivated by the fact that elements of the affine part $\mathcal{C}_Q(5)$ are also curves

in \mathbb{P} . The intersection of the curve C with S is the same as the intersection with $S_0 = S \cap \{z = 0\}$, and C intersects S_0 with multiplicity at least 5 at Q .

Suppose S_0 is smooth. Then S_0 has genus 1, so C has no components in common with S_0 . The curves S_0 and C also have no components in common if $C \in \overline{\mathcal{C}}_Q(5) - (\mathcal{C}_Q(5) \cup \mathcal{C}'_Q(5))$ (Lemma 3.6 and Remark 3.7) or $C = C_2$ as in Lemma 3.4. Hence, in all these cases there is a unique sixth intersection point in $C \cap S = C \cap S_0$, and we can extend σ to C by sending C to this sixth intersection point, say R ; the divisor $5(Q) + (R)$ on S_0 is a hypersurface section inside the plane given by $z = 0$, so it is linearly equivalent to a multiple of $3(\mathcal{O} \cap S_0)$ on S_0 , and we find $R = -5Q$ in $S_0^{\text{ns}}(k)$.

We have reduced to the case that S_0 has a cusp (Lemma 3.5), or S_0 has a node and $C = C_1$ as in Lemma 3.4. In both cases, there is a $d \in k$ such that $f_0 = -3d^2$ and $g_0 = 2d^3$ and $C = (\alpha, 0) \in \mathcal{C}'_Q(5) \subset \mathbb{A}^2(q', r)$ with $\alpha = \frac{1}{4}(x_0 + 2d)^{-1}$. The functions $F'_i = r^i F_i$ on $\overline{\mathcal{C}}_Q(5)$ are regular at C and in fact, we have

$$\begin{aligned} F'_5 &= -2b'c' + 3q'^2 + f_4r^4 + f_3q'r^3 + g_5r^5, \\ F'_6 &= -c'^2 + q'^3 + f_4q'r^4 + g_6r^6. \end{aligned}$$

On $\mathcal{C}'_Q(5)$, the rational map $\rho: \mathcal{C}'_Q(5) \rightarrow \mathbb{P}^1(z', w')$ is given by

$$\rho(C) = (C, [-F_5(C) : r(C)F_6(C)]) = (C, [-F'_5(C) : F'_6(C)]).$$

The functions r and $q' - \alpha$ generate the maximal ideal \mathfrak{m} of the local ring A_C at C . In A_C we have

$$(11) \quad c_1q'^2 + c_2q' - c_5 \equiv (c_6 - q'c_3)r$$

modulo r^2 .

Now suppose $\phi_3 \neq 0$. Then $c_1 \neq 0$ and the left-hand side of (11) factors as $c_1(q' - \alpha)(q' - \alpha')$ with $\alpha' = \frac{1}{4}(x_0 + 7d)(x_0 + 2d)^{-1}(x_0 + 3d)^{-1}$. Assume $d \neq 0$ as well. Then $\alpha' \neq \alpha$, so modulo \mathfrak{m}^2 we have $q' - \alpha \equiv \delta r$ with

$$\delta = \frac{c_6 - c_3\alpha}{c_1(\alpha - \alpha')}.$$

Hence, \mathfrak{m} is generated by r and one checks, preferably by computer, that we have

$$(12) \quad F'_5 \equiv \frac{(f_1d + g_1)\phi_5}{(x_0 - d)^{10}\phi_2^2} \cdot r \quad \text{and} \quad F'_6 \equiv \frac{(f_1d + g_1)\phi_4}{(x_0 - d)^5\phi_2^2} \cdot r \quad (\text{mod } r\mathfrak{m}).$$

We claim that (12) also holds when $d = 0$ or $\phi_3 = 0$. Indeed, if $\phi_3 = 0$, then one uses $x_0 = -3d$, while $c_1 = 0$ and $c_2 \neq 0$, so (11) yields $q' - \alpha \equiv c_2^{-1}(c_6 - \alpha c_3)r \pmod{\mathfrak{m}^2}$ and thus r generates \mathfrak{m} ; if $d = 0$, then \mathfrak{m} may not be principal, but using that modulo $r\mathfrak{m}$ we have (11) and $q'r \equiv \alpha r$, it is easy to check that (12) holds. Hence, (12) holds in all cases.

Now $f_1d + g_1$ is nonzero because the surface S is smooth at the singular point of S_0 . Also, since Q is not the singular point of S_0 , we have $x_0 \neq d$ and ϕ_4 and ϕ_5 do not both vanish. We conclude that $\rho: \mathcal{C}'_Q(5) \rightarrow \mathbb{P}^1(z', w')$ is well defined at C , sending C to $(C, [-\phi_5 : (x_0 - d)^5\phi_4])$. Substituting this into the parametrization of Lemma 3.3, it is not hard to check that $\sigma(C) = \gamma(\rho(C)) = -4Q$ in $S_0^{\text{ns}}(k)$, though a computer is again quite useful. \square

4. EXAMPLES

In this section, k still denotes a field of characteristic not equal to 2 or 3. We will give examples of surfaces $S \subset \mathbb{P}$ over k given by (1), together with a point $Q \in S(k)$ for which the map $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$ sends at least one irreducible component of $\overline{\mathcal{C}}_Q(5)$ to a fiber of $\varphi|_S: S \rightarrow \mathbb{P}^1$. In the next section we will see that, at least outside characteristic 5, these examples include all cases where every component of $\overline{\mathcal{C}}_Q(5)$ is sent to a fiber on S .

Example 4.1. Let $\beta, \delta \in k^*$ and assume the characteristic of k is not 5. Set

$$\begin{aligned} x_0 &= 3(\beta^2 + 6\beta + 1), & f_0 &= -27(\beta^4 + 12\beta^3 + 14\beta^2 - 12\beta + 1), & f &= f_0w^4, \\ y_0 &= 108\beta, & g_0 &= 54(\beta^2 + 1)(\beta^4 + 18\beta^3 + 74\beta^2 - 18\beta + 1), & g &= \delta z^5w + g_0w^6, \end{aligned}$$

and let $S \subset \mathbb{P}$ be the surface given by (1) with point $Q = [x_0 : y_0 : 0 : 1]$. Assume that S is smooth, so that it is a del Pezzo surface. The curve S_0 is nonsingular if and only if $\beta(\beta^2 + 11\beta - 1) \neq 0$; the point Q has order 5 in $S_0^{\text{ns}}(k)$. Generically, in particular over a field in which β and δ are independent transcendentals, the surface S is smooth and the fibration $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ has 10 nodal fibers (type I_1) and one cuspidal fiber (type II) above $[z : w] = [1 : 0]$.

Let α be an element in a field extension of k satisfying $\alpha^2 = \alpha + 1$. Then $\overline{\mathcal{C}}_Q(5)$ splits over $k(\alpha)$ into two components. The function F_6 vanishes on $\overline{\mathcal{C}}_Q(5)$ and the map $\sigma: \mathcal{C}_Q(5) \rightarrow S$ sends each component birationally to the cuspidal fiber. Both components of T intersect S in the cuspidal fiber and, over an extension of $k(\alpha)$ of degree at most 5, five (-1) -curves; the surface T intersects S doubly in the cuspidal curve, as well as in ten (-1) -curves going through Q , corresponding to the points on the affine part $\mathcal{C}_Q(5)$ where F_5 vanishes. Indeed, if α, ϵ in an extension of k satisfy

$$\alpha^2 = \alpha + 1 \quad \text{and} \quad \delta = -6(\beta + \alpha^5)\epsilon^5,$$

then we have a section over $k(\alpha, \epsilon)$ going through Q with

$$\begin{aligned} x &= \epsilon^2 z^2 + 6\alpha \epsilon z w + x_0 w^2, \\ y &= -\epsilon^3 z^3 + 3(\beta + 2\alpha + 3)\epsilon^2 z^2 w + 18\alpha(\beta + 1)\epsilon z w^2 + y_0 w^3. \end{aligned}$$

Example 4.2. For any $\beta \neq 0$, the point $(x_0, y_0) = (3, \beta)$ has order 3 on the Weierstrass curve given by $y^2 = x^3 + f_0 x + g_0$ with $f_0 = 6\beta - 27$ and $g_0 = \beta^2 - 18\beta + 54$; this curve is nonsingular if and only if $\beta \neq 4$.

Subexample (i). For any $\alpha_1, \alpha_2, \alpha_3 \in k$ we consider the surface $S \subset \mathbb{P}$ given by (1) with

$$\begin{aligned} f &= -3\alpha_1^2 z^4 + 3\alpha_2 z^3 w + (18 - 3\beta)\alpha_1 z^2 w^2 + f_0 w^4, \\ g &= \alpha_3 z^6 + 3\alpha_1 \alpha_2 z^5 w + (18 - 6\beta)\alpha_1^2 z^4 w^2 + (\beta - 9)\alpha_2 z^3 w^3 + (15\beta - 54)\alpha_1 z^2 w^4 + g_0 w^6, \end{aligned}$$

and with $Q = [3 : \beta : 0 : 1]$, so that Q has order 3 on $S_0^{\text{ns}}(k)$. Assume S is smooth, so that it is a del Pezzo surface. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by

$$(p^2 - \beta\alpha_1)(\beta q - p^2 + 2\beta\alpha_1) = 0.$$

The function $F_5 = 3\beta^{-1}p(q + \alpha_1)(\beta q - p^2 + 2\beta\alpha_1)$ vanishes on the component given by the vanishing of the second factor; this component is contracted by the map $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$, which sends it to Q . There are six (-1) -curves on S going through Q .

Subexample (ii). For any $\alpha_4, \alpha_5, \alpha_6 \in k$ we consider the surface $S \subset \mathbb{P}$ given by (1) with

$$\begin{aligned} f &= 3\alpha_4 z^3 w + f_0 w^4, \\ g &= \alpha_6 z^6 + \alpha_5 z^3 w^3 + g_0 w^6, \end{aligned}$$

and with $Q = [3 : \beta : 0 : 1]$. Assume S is smooth, so that it is a del Pezzo surface. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by

$$p(\beta pq - p^3 + (\beta - 9)\alpha_4 - \alpha_5) = 0.$$

Again, the function $F_5 = 3\beta^{-1}q(\beta pq - p^3 + (\beta - 9)\alpha_4 - \alpha_5)$ vanishes on the component given by the vanishing of the second factor; this component is contracted by the map $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$, which sends it to Q . There are nine (-1) -curves on S going through Q .

Subexample (iii). Let S be any smooth surface that fits in both families of these examples, i.e., with $\alpha_1 = 0$, $\alpha_4 = \alpha_2$, $\alpha_5 = (\beta - 9)\alpha_2$, and $a_6 = a_3$. Writing $\epsilon = a_2$ and $\delta = a_3$, we have

$$\begin{aligned} f &= 3\epsilon z^3 w + f_0 w^4, \\ g &= \delta z^6 + (\beta - 9)\epsilon z^3 w^3 + g_0 w^6. \end{aligned}$$

Generically, say over a field in which β, δ , and ϵ are independent transcendentals, the surface S is smooth and the fibration $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ has twelve nodal fibers. Suppose S is indeed smooth. Then $\beta \notin \{0, 4\}$. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by

$$p^2(\beta q - p^2) = 0,$$

so it consists of two components. Both components are contracted by $\sigma: \mathcal{C}_Q(5) \rightarrow S$, which maps these components to Q . There are nine (-1) -curves on S going through Q .

Example 4.3. For any $\beta \in k^*$, the point $(0, \beta)$ has order 3 on the elliptic curve given by $y^2 = x^3 + \beta^2$. In the following three subexamples, we will take $g = \epsilon z^6 + \delta z^3 w^3 + \beta^2 w^6$ for some $\delta, \epsilon \in k$ and the point $Q = [0 : \beta : 0 : 1] \in \mathbb{P}$, which in all cases will have order 3 on S_0 .

Subexample (i). Let S be the surface given by (1) with $f = \alpha z^2 w^2$ for some $\alpha \in k$ and assume that S is smooth. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by $(3p^2 + \alpha)q = 0$. The function $F_5 = 3pq^2$ vanishes on the component given by $q = 0$; this component is contracted by the map $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$, which sends it to Q . There are six (-1) -curves on S going through Q . Generically, there are twelve nodal fibers.

Subexample (ii). Let S be the surface given by (1) with $f = \alpha z^3 w$ for some $\alpha \in k$ and assume that S is smooth. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by $p(3pq + \alpha) = 0$. The function $F_5 = q(3pq + \alpha)$ vanishes on one of the components; this component is contracted by the map $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$, which sends it to Q . There are nine (-1) -curves on S going through Q . Generically, there are twelve nodal fibers.

Subexample (iii). Let S be the surface given by (1) with $f = 0$ and assume that S is smooth. The affine part $\mathcal{C}_Q(5)$ of the curve $\overline{\mathcal{C}}_Q(5)$ is given by $p^2 q = 0$ and we have $\sigma(\overline{\mathcal{C}}_Q(5)) = Q$. The surface is isotrivial; all fibers have j -invariant 0. There are nine (-1) -curves on S going through Q , and there are six cuspidal fibers.

5. A MULTISECTION

We continue the notation of Sections 2 and 3. In particular, the field k , the surface S , and the point Q are fixed as before, as are all the objects that depend on them.

As we have seen in the previous section, not every component of $\overline{\mathcal{C}}_Q(5)$ necessarily has its image under $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$ map dominantly to \mathbb{P}^1 under the projection $\varphi|_S: S \rightarrow \mathbb{P}^1$. Proposition 5.2 states that this does hold for every component if the order of Q is larger than 6. Indeed, there are examples where the order of Q is 6 and $\mathcal{C}_Q(5)$ has a component that maps under σ to Q .

Lemma 5.1. *Let \mathcal{C}_0 be a component of $\overline{\mathcal{C}}_Q(5)$ for which $\varphi(\sigma(\mathcal{C}_0)) = [0 : 1]$. Then $\sigma(\mathcal{C}_0) = Q$.*

Proof. Without loss of generality, we assume k is algebraically closed. Consider any section $C \in \mathcal{C}_0 \cap \mathcal{C}_Q(5)$ that is not entirely contained in S . Then σ is well defined at C and $\sigma(C)$ is the unique sixth intersection point of C with S . Since C is a section of $\varphi: S \rightarrow \mathbb{P}^1$, it intersects the fiber S_0 only once, namely in Q , and as this sixth intersection point lies in S_0 as well, we conclude $\sigma(C) = Q$. Thus all but finitely many points of \mathcal{C}_0 map to Q under σ , so $\sigma(\mathcal{C}_0) = Q$. \square

Proposition 5.2. *Suppose the order of Q in $S_0^{\text{ns}}(k)$ is larger than 5 and $\overline{\mathcal{C}}_Q(5)$ has a component \mathcal{C}_0 that maps under $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$ to a fiber of φ . Then Q has order 6 and $\sigma(\mathcal{C}_0) = Q$. The curve $\overline{\mathcal{C}}_Q(5)$ has a unique second component, which is sent under σ to a horizontal curve on S .*

Proof. Since \mathcal{C}_0 is projective, it contains a point in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$, say R , which by Proposition 3.9 is mapped under σ to $-4Q$ or $-5Q$ on S_0 . As the order of Q is at least 6, this image is not equal to \mathcal{O} , so the composition of σ with $\varphi: S \rightarrow \mathbb{P}^1$ sends R to $[0 : 1] \in \mathbb{P}^1$. Suppose σ does not send \mathcal{C}_0 to a horizontal curve. Then the composition $\varphi \circ \sigma$ sends \mathcal{C}_0 to $[0 : 1]$. From Lemma 5.1 we conclude $\sigma(\mathcal{C}_0) = Q$ and we obtain $Q = \sigma(R) = -4Q$ or $Q = \sigma(R) = -5Q$. As the order of Q is larger than 5, we find that the order is 6 and $\sigma(R) = -5Q$.

With equations (7) one checks easily that $c_2^2 + 4c_1c_5$ equals $\phi_2^2(\phi_4^2 - 4\phi_6)$, and as $\phi_6 = 0$ (together with $y_0 \neq 0$) implies $\phi_4 \neq 0$, we get $c_2^2 + 4c_1c_5 \neq 0$, which in turn, together with $c_1 \neq 0$, implies that $\mathcal{C}_Q(5)$ is reduced. Suppose that each component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a fiber of φ . Then as above, we find $(\varphi \circ \sigma)(\overline{\mathcal{C}}_Q(5)) = [0 : 1]$ and as the composition $\varphi \circ \sigma$ is given by $[-F_5 : F_6]$, we find that F_5 vanishes on $\mathcal{C}_Q(5)$; as $\mathcal{C}_Q(5)$ is reduced, this implies that if we view F_4 and F_5 as polynomials in $k[p, q]$ (cf. Remark 2.5), then F_5 is a multiple of F_4 . Comparing the coefficients

in $k[p]$ of q^2 in

$$\begin{aligned}\phi_2^3 F_4 &= \phi_2^2 \phi_3 q^2 + (-3\phi_2 \phi_4 p^2 + \dots)q + \dots, \\ \phi_2^3 F_5 &= \phi_2((\phi_2^2 - 2\phi_4)p - \psi l_1)q^2 + ((\phi_4 \psi - 4\phi_3^2)p^3 + \dots)q + \dots,\end{aligned}$$

we find

$$\phi_2 \phi_3 F_5 = ((\phi_2^2 - 2\phi_4)p - \psi l_1)F_4.$$

Comparing the coefficient of $p^3 q$ in this equality gives

$$\phi_3(\phi_4 \psi - 4\phi_3^2) = -3\phi_4(\phi_2^2 - 2\phi_4),$$

With equations (7) we find that this is equivalent to $4\phi_6 = \phi_4^2$, so we obtain $\phi_4 = \phi_6 = 0$, a contradiction from which we conclude that not all components map to a vertical component. It follows that there is a second component, which is unique as $c_1 \neq 0$ implies that there are at most two components, and this component maps to a horizontal curve on S . \square

We say that two pairs (S_1, Q_1) and (S_2, Q_2) of a surface with a point on it are isomorphic if there is an isomorphism from S_1 to S_2 that maps Q_1 to Q_2 . For example, the involution $\iota: \mathbb{P} \rightarrow \mathbb{P}$ that sends $[x : y : z : w] \in \mathbb{P}$ to $[x : y : -z : w - z]$ fixes Q , so it induces an isomorphism, also denoted ι , from the pair (S, Q) to $(\iota(S), Q)$; the surface $\iota(S)$ is given by $y^2 = x^3 + \tilde{f}(z, w)x + \tilde{g}(z, w)$, with

$$\begin{aligned}\tilde{f}(z, w) &= f(-z, w - z) = f_0 w^4 + (-4f_0 - f_1)w^3 z + \dots, \\ \tilde{g}(z, w) &= g(-z, w - z) = g_0 w^6 + (-6g_0 - g_1)w^5 z + \dots.\end{aligned}$$

Note that ι fixes the points in the fiber above $[0 : 1]$ and it switches the fibers above $[1 : 1]$ and $[1 : 0]$. It also fixes f_0 and g_0 and it replaces f_1 and g_1 by $-4f_0 - f_1$ and $-6g_0 - g_1$, respectively.

Proposition 5.3. *Suppose that the characteristic of k is not 2, 3, or 5, and $5Q = \mathcal{O}$ in $S_0^{\text{ns}}(k)$. If no component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a horizontal curve on S , then there exist $\beta, \delta \in k$ such that the pair (S, Q) is isomorphic to the pair of Example 4.1.*

Proof. As Q has order 5, we have $\phi_5 = 0$ and $\phi_3, c_1 \neq 0$ and there are $\beta, \eta \in k^*$ such that

$$\begin{aligned}x_0 &= 3(\beta^2 + 6\beta + 1)\eta^2, \\ y_0 &= 108\beta\eta^3, \\ f_0 &= -27(\beta^4 + 12\beta^3 + 14\beta^2 - 12\beta + 1)\eta^4, \\ g_0 &= 54(\beta^2 + 1)(\beta^4 + 18\beta^3 + 74\beta^2 - 18\beta + 1)\eta^6.\end{aligned}$$

Without loss of generality, we assume $\eta = 1$. The fiber S_0 is singular if and only if $D = \beta(\beta^2 + 11\beta - 1) = 0$, and in this case S_0 is nodal. Suppose no component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a horizontal curve on S , so $\varphi \circ \sigma$ has finite image. Note that there is a component \mathcal{C}_1 with $\varphi(\sigma(\mathcal{C}_1)) \neq [0 : 1]$, because if we had $\varphi(\sigma(\overline{\mathcal{C}}_Q(5))) = [0 : 1]$, then we would have $\sigma(\overline{\mathcal{C}}_Q(5)) = Q = -4Q$ by Lemma 5.1; by Lemma 3.4 and Proposition 3.9, the fiber S_0 would be cuspidal, as $\phi_3 \neq 0$, contradiction. Without loss of generality, we assume $\varphi(\sigma(\mathcal{C}_1)) = [1 : 0]$ and we write $S_\infty = \varphi^{-1}([1 : 0])$.

Assume that there is a second component \mathcal{C}_0 of $\overline{\mathcal{C}}_Q(5)$ with $t_0 \neq [1 : 0] = \infty$, where we write $t_0 = \varphi(\sigma(\mathcal{C}_0))$. Since $\overline{\mathcal{C}}_Q(5)$ has at most two components and both are reduced if there are two, its components are \mathcal{C}_0 and \mathcal{C}_1 . We first consider the case $t_0 = [0 : 1]$. From Lemma 5.1 we find $\sigma(\mathcal{C}_0) = Q = -4Q$, and as \mathcal{C}_0 contains points of Ω , we conclude that S_0 is singular from Proposition 3.9, so $\beta^2 + 11\beta - 1 = 0$. If we consider F_4, F_5 , and F_6 as polynomials in q over $k[p]$ (cf. Remark 2.5), then F_5 and F_6 vanish on \mathcal{C}_0 and \mathcal{C}_1 , respectively, so F_4 divides $F_5 F_6$. Since the main coefficient c_1 of F_4 as a polynomial in q over $k[p]$ is invertible, we can compute (by computer, with $x_0, f_0, \dots, f_4, g_0, \dots, g_6$ independent transcendentals) the remainder of $F_5 F_6$ upon division by F_4 , which is a polynomial $L = \mu q + \nu$, with $\mu, \nu \in k[p]$ of degree 9 and 11, respectively. Our special values of x_0, f_0, g_0 already imply that the coefficients of p^{11} and $p^9 q$ in L specialize to 0, and the fact that F_4 divides $F_5 F_6$ implies that L specializes to 0. Assume for the remainder of this paragraph that the characteristic of k is not 11, 17, 23, or 29. Then the vanishing of the (specialization of the) coefficients of $p^8 q, p^7 q, p^6 q$, and $p^5 q$ in L determine, in that order, the values

of f_1, f_2, f_3 , and f_4 in terms of g_0, \dots, g_6 . The vanishing of the coefficient of p^8 then implies $g_1 = 0$ or $g_2 = \lambda g_1^2$ for some specific constant λ . In the case $g_1 = 0$, the vanishing of the coefficient of p^7 yields $g_2 = 0$; then the vanishing of the coefficients of p^5 and p^3q implies $g_3 = g_6 = 0$, and finally the vanishing of the coefficients of p^3 gives $g_4 = 0$, which shows that the pair (S, Q) is isomorphic to the pair in Example 4.1, with $\delta = g_5$, though F_6 vanishes on both components and S is singular. In the case $g_2 = \lambda g_1^2$ we may assume $g_1 \neq 0$, and the vanishing of the coefficients of p^7, p^6, p^5 , and finally p^3q , express g_3, g_4, g_6, g_5 , in that order, in terms of the remaining unknown coefficients of g , which in the end yields a surface S that is singular, so this case does not occur.

If the characteristic of k is equal to 11, 17, 23, or 29, and in fact whenever the characteristic is not equal to 2, 3, 5, 7, 13, or 19, we similarly solve for the parameters $f_1, \dots, f_4, g_1, \dots, g_6$, except that we start by expressing g_1, \dots, g_4 in terms of f_1, \dots, f_4 . We conclude also in these characteristics that S is singular.

Still assuming there is a second component \mathcal{C}_0 of $\overline{\mathcal{C}}_Q(5)$ with $t_0 \neq [1 : 0] = \infty$, we now consider the case that $t_0 \neq [0 : 1], [1 : 0]$. Then after applying an automorphism of the base curve \mathbb{P}^1 that fixes $[0 : 1]$ and $[1 : 0]$, we may assume $t_0 = [1 : 1]$, so that $F_5 + F_6$ and F_6 vanish on \mathcal{C}_0 and \mathcal{C}_1 , respectively, and the product $(F_5 + F_6)F_6$ is divisible by F_4 . Note that ι switches the fibers above t_0 and t_1 , so without loss of generality we may still apply ι at some point. Since the points in $\Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$ map under σ to S_0 , and $\varphi(S_0 - \{\mathcal{O}\}) = [0 : 1]$, the points in Ω map under σ to $\mathcal{O} = -5Q$; it follows from Proposition 3.9 that S_0 is smooth, so we find $D \neq 0$ and in particular $\beta \neq 0$. We proceed as before. Viewing F_4, F_5, F_6 as polynomials in q over $k[p]$, we find that generically, say over a field in which $x_0, f_0, \dots, f_4, g_0, \dots, g_6$ are independent transcendentals, there are $d_0, \dots, d_{10}, e_0, \dots, e_{12}$, in terms of these transcendentals, such that

$$(F_5 + F_6)F_6 \equiv (d_{10}p^{10} + \dots + d_1p + d_0)q + e_{12}p^{12} + \dots + e_1p + e_0 \pmod{F_4}.$$

The fact that Q has order 5 implies $d_{10}, d_9, e_{12}, e_{11}$ specialize to 0. In our case, the other coefficients $d_0, \dots, d_8, e_0, \dots, e_{10}$ specialize to 0 as well. We claim that from the fact that e_{10} and d_8 specialize to 0, it follows that

$$(13) \quad \begin{cases} f_1 = 0 \\ g_1 = 0 \end{cases} \quad \text{or} \quad \begin{cases} f_1 = -4f_0 \\ g_1 = -6g_0 \end{cases} \quad \text{or} \quad \begin{cases} f_1 = -2f_0 - 54\gamma^{-1}\lambda \\ g_1 = -3g_0 + 54\gamma^{-1}\mu \end{cases}$$

for some element $\gamma \in k$ with $\gamma^2 = 5$ and with

$$\begin{aligned} \lambda &= (\beta^2 + 1)(\beta^2 + 10\beta - 1), \\ \mu &= 3(\beta^6 + 16\beta^5 + 49\beta^4 - 40\beta^3 - 49\beta^2 + 16\beta - 1). \end{aligned}$$

Indeed, for any γ in an extension of k with $\gamma^2 = 5$ and $\omega = \frac{1}{2}(3 - \gamma)$, the linear combinations

$$\begin{aligned} & \frac{1}{2^5 3^3} \phi_2^4 ((3\beta - 1)(7\beta + 1)d_8 + 180\beta(11\beta - 2)e_{10}) \\ &= (3g_0f_1 - 2f_0g_1) \cdot ((f_1 + 2f_0)\mu + (g_1 + 3g_0)\lambda) \quad \text{and} \\ & \frac{1}{4} \phi_2^4 (\omega^4 \beta^2 + (2\gamma - 4)\beta + \omega^{-1}) (d_8 + 36\omega e_{10}) \\ &= (\gamma(3g_0f_1 - 2f_0g_1) - 54(g_1\lambda + f_1\mu)) \cdot (\gamma(3g_0f_1 - 2f_0g_1) - 54((g_1 + 6g_0)\lambda + (f_1 + 4f_0)\mu)) \end{aligned}$$

of d_8 and e_{10} factor into two linear factors. Therefore, the vanishing of d_8 and e_{10} implies the vanishing of one of the first two factors and one of the second two. The four combinations give four systems of two linear equations in the two variables f_1 and g_1 . For each combination, the determinant of the system is a nonzero multiple of D and therefore nonzero itself. The systems yield exactly the four claimed pairs for (f_1, g_1) in (13).

Note that as the isomorphism ι replaces f_1 and g_1 by $-4f_0 - f_1$ and $-6g_0 - g_1$, respectively, it switches the first two cases in (13), as well as the last two cases given by the third pair for $\pm\gamma$. Therefore, it suffices to finish the proof for the two subcases $(f_1, g_1) = (0, 0)$ and $(f_1, g_1) = (-2f_0 - 54\gamma^{-1}\lambda, -3g_0 + 54\gamma^{-1}\mu)$. Suppose first we are in the former subcase, so $(f_1, g_1) = (0, 0)$. Then the equations $d_7 = e_9 = 0$ determine a system of two linear equations in f_2 and g_2 , of which the determinant is a nonzero multiple of D and therefore nonzero itself. The unique solution is

$f_2 = g_2 = 0$. Subsequently, the system $d_6 = e_8 = 0$ gives $f_3 = g_3 = 0$ and then the system $d_5 = e_7 = 0$ yields $f_4 = g_4 = 0$. At this point, the coefficients d_4 and e_6 specialize to 0 automatically, and the equation $d_3 = 0$ determines $g_6 = 0$. With $g_5 = \delta$, we obtain exactly the surface of Example 4.1.

Now suppose we are in the latter subcase, so $(f_1, g_1) = (-2f_0 - 54\gamma^{-1}\lambda, -3g_0 + 54\gamma^{-1}\mu)$. As in the previous subcase, the linear systems $d_{9-i} = e_{11-i} = 0$ determine f_i and g_i inductively for $i = 2, 3, 4$. Again, the coefficients d_4 and e_6 then specialize to 0 automatically. Finally, the system $d_3 = e_6 = 0$ is linear in g_5 and g_6 and determines these two parameters uniquely. However, this yields a surface S that is singular. More specifically, the associated minimal elliptic surface has two singular fibers of type I_5 . This contradiction finishes the proof of this case.

What remains is the case that there is no component \mathcal{C}_0 of $\overline{\mathcal{C}}_Q(5)$ with $\varphi(\sigma(\mathcal{C}_0)) \neq \infty$. Then we have $\sigma(\overline{\mathcal{C}}_Q(5)) \subset S_\infty$, so F_6 vanishes on $\mathcal{C}_Q(5)$, and also $\sigma(\Omega) \subset S_\infty$. From Proposition 3.9 we conclude $\sigma(\Omega) \subset S_0 \cap S_\infty = \{\mathcal{O}\} = \{-5Q\}$; it follows from Proposition 3.9 that S_0 is smooth, so we find $D \neq 0$ and in particular $\beta \neq 0$. With equations (7) one checks easily that $c_2^2 + 4c_1c_5$ equals $\phi_2^2(5\phi_4^2 - 4\phi_5)$, and as $\phi_5 = 0$ (together with $y_0 \neq 0$) implies $\phi_4 \neq 0$, we get $c_2^2 + 4c_1c_5 \neq 0$, which in turn, together with $c_1 \neq 0$, implies that $\mathcal{C}_Q(5)$ is reduced. Therefore, F_6 is a multiple of F_4 , so this is a subcase of both cases above, where F_5F_6 and $(F_5 + F_6)F_6$ were multiples of F_4 , and we are done. \square

Indeed, in characteristic 5, there are other examples than those mentioned in Proposition 5.3 where Q has order 5 and no component of $\overline{\mathcal{C}}_Q(5)$ maps under σ to a horizontal curve on S . It takes less computational force to deal with the case that Q has order 4.

Proposition 5.4. *Suppose $4Q = \mathcal{O}$ in $S_0^{\text{ns}}(k)$. Then $\mathcal{C}_Q(5)$ has a component that maps under $\sigma: \mathcal{C}_Q(5) \rightarrow S$ to a horizontal curve on S .*

Proof. First note that the fiber S_0 does not have a cusp, as the additive reduction together with the identity $4Q = \mathcal{O}$ would imply that the characteristic of k is 2, which it is not by assumption. Therefore, by Proposition 3.9, at least one of the points in Ω maps to $-5Q = -Q$. Let R be such a point and let \mathcal{C}_0 be a component of $\overline{\mathcal{C}}_Q(5)$ that contains R . Suppose that \mathcal{C}_0 is sent by σ to a fiber on S , so that $\varphi(\sigma(\mathcal{C}_0))$ is a point on \mathbb{P}^1 . From $\sigma(R) = -Q \in S_0$ we conclude $\varphi(\sigma(\mathcal{C}_0)) = [0 : 1]$, and Lemma 5.1 implies $\sigma(\mathcal{C}_0) = Q$, which contradicts $\sigma(R) = -Q$, so \mathcal{C}_0 is sent to a horizontal curve on S . \square

Finally, we deal with the case that Q has order 3.

Proposition 5.5. *Suppose $3Q = \mathcal{O}$ in $S_0^{\text{ns}}(k)$. Then $\mathcal{C}_Q(5) \subset \mathbb{A}^2(p, q)$ has a unique component that projects birationally to $\mathbb{A}^1(p)$. If this component maps under $\sigma: \mathcal{C}_Q(5) \rightarrow S$ to a vertical curve on S , then the pair (S, Q) is isomorphic to one of the pairs described in Examples 4.2 and 4.3. Moreover, if another component of $\mathcal{C}_Q(5)$ maps to a horizontal curve on S as well, then the pair (S, Q) is isomorphic to one of the pairs described in Subexamples 4.2(iii) and 4.3(iii).*

Proof. Any component \mathcal{C} of $\overline{\mathcal{C}}_Q(5)$ contains a point in $R \in \Omega = \overline{\mathcal{C}}_Q(5) - \mathcal{C}_Q(5)$, which satisfies $\sigma(R) = -4Q$ or $\sigma(R) = -5Q$ by Proposition 3.9; as $-4Q$ and $-5Q$ do not equal \mathcal{O} , we find $\varphi(\sigma(R)) = [0 : 1]$, so if $\sigma(\mathcal{C})$ is contained in a fiber of $S \rightarrow \mathbb{P}^1$, then F_5 vanishes on \mathcal{C} .

As the order of Q is 3, we have $\phi_3 = 0$ and thus $\phi_2\phi_4 \neq 0$, so $c_1 = 0$ and $c_2 \neq 0$, and the curve $\mathcal{C}_Q(5)$ is given by $mq = n$ with

$$m = c_2p^2 + c_3p + c_4 \quad \text{and} \quad n = c_5p^4 + c_6p^3 + c_7p^2 + c_8p + c_9.$$

From $c_2 \neq 0$, we find that m is not identically 0, so indeed, there is a unique component of $\mathcal{C}_Q(5)$, say \mathcal{C}_1 , that projects birationally to $\mathbb{A}^1(p)$. Assume that $\sigma(\mathcal{C}_1)$ is contained in a fiber on S . Then F_5 vanishes on \mathcal{C}_1 by the above. If we write F_5 as $F_5 = \delta_1q^2 + \delta_2q + \delta_3$, with $\delta_i \in k[p]$ of degree $2i - 1$, then we find

$$(14) \quad \delta_1n^2 + \delta_2mn + \delta_3m^2 = 0.$$

Let L denote the left-hand side of (14). A priori, say over a field in which $x_0, f_0, \dots, f_4, g_0, \dots, g_6$ are independent transcendentals, the polynomial $L \in k[p]$ has degree 9, but from $\phi_3 = 0$, it already

follows that the degree is at most 8. We will use the vanishing of all coefficients to identify the pair (S, Q) .

As Q has order 3, there are $\beta, \eta \in k^*$ such that

$$\begin{array}{ll} x_0 = 3\eta^2, & x_0 = 0, \\ y_0 = \beta\eta^3, & y_0 = \beta, \\ f_0 = (6\beta - 27)\eta^4, & f_0 = 0, \\ g_0 = (\beta^2 - 18\beta + 54)\eta^6, & g_0 = \beta^2. \end{array} \quad \text{or}$$

We start with the first case and without loss of generality, we assume $\eta = 1$. The vanishing of the coefficient of p^8 in L gives $3f_1g_0 = 2f_0g_1$. Since f_0 and g_0 do not both vanish, there is a $\delta \in k$ such that $f_1 = 2\delta f_0$ and $g_1 = 3\delta g_0$. After applying an automorphism of \mathbb{P}^1 given by $[z : w] \mapsto [2z : \delta z + 2w]$, we may assume without loss of generality that $\delta = 0$, so $f_1 = g_1 = 0$. Then the vanishing of the coefficient of p^7 in L shows that there is a $\alpha_1 \in k$ such that $f_2 = (18 - 3\beta)\alpha_1$ and $g_2 = (15\beta - 54)\alpha_1$. The coefficient of p^6 now vanishes automatically and the vanishing of the coefficient of p^5 yields $g_4 = (\beta - 3)(f_4 - 3\alpha_1^2)$. Subsequently, the vanishing of the coefficient of p^4 gives $g_5 = ((2\beta - 9)f_3 - 3g_3)\alpha_1\beta^{-1}$. Then the coefficient of p^3 vanishes automatically. The vanishing of the coefficient of p^2 yields $f_4 = -3\alpha_1^2$ or $3g_3 = (\beta - 9)f_3$, but in the latter case, the vanishing of the coefficient of p yields the former, so we have $f_4 = -3\alpha_1^2$ in any case. Finally, the vanishing of the coefficient of p gives $3g_3 = (\beta - 9)f_3$ or $\alpha_1 = 0$; the former case yields Example 4.2(i) with $\alpha_2 = \frac{1}{3}f_3$ and $\alpha_3 = g_6$, while the latter case yields Example 4.2(ii) with $\alpha_4 = \frac{1}{3}f_3$, $\alpha_5 = g_3$ and $\alpha_6 = g_6$.

Suppose we are in the former case, so the case of Example 4.2(i). If σ sends one of the components of $\mathcal{C}_Q(5)$ given by $p^2 - \beta\alpha_1 = 0$ to a fiber of φ , then the first argument of this proof shows that $\frac{1}{3}\beta F_5 = p(q + \alpha_1)(\beta q - p^2 + 2\beta\alpha_1)$ vanishes on this component, which implies $\alpha_1 = 0$, so the pair (S, Q) belongs to the family described in Example 4.2(iii). Now suppose we are in the latter case, so the case of Example 4.2(ii). If σ sends the component of $\mathcal{C}_Q(5)$ given by $p = 0$ to a fiber of φ , then similarly $F_5 = 3\beta^{-1}q(\beta pq - p^3 + (\beta - 9)\alpha_4 - \alpha_5)$ vanishes on this component, which implies $\alpha_5 = (\beta - 9)\alpha_4$, so again the pair (S, Q) belongs to the family described in Example 4.2(iii).

We now consider the second case, so with $(x_0, y_0, f_0, g_0) = (0, \beta, 0, \beta^2)$. Since $g_0 \neq 0$, we may apply an automorphism of $\mathbb{P}^1(z, w)$ given by $[z : w] \mapsto [6g_0z : g_1z + 6g_0w]$ to reduce to the case $g_1 = 0$. The vanishing of the coefficients of p^8, p^7, p^5 , and p^4 in L yields $f_1 = g_2 = f_4 = g_5 = 0$. Then the vanishing of the coefficient of p^2 yields $f_3g_4 = 0$; if $f_3 = 0$, then the vanishing of the coefficient of p in L gives $g_4 = 0$, so we have $g_4 = 0$ in any case. Then the vanishing of the coefficient of p gives $f_2 = 0$ or $f_3 = 0$ and these cases correspond to Subexamples 4.3(ii) and 4.3(i), respectively.

Suppose we are in the case of Example 4.3(i), so $f_3 = 0$ and $f_2 = \alpha$. If σ sends one of the components of $\mathcal{C}_Q(5)$ given by $3p^2 + \alpha = 0$ to a fiber of φ , then the first argument of this proof shows that $F_5 = 3pq^2$ vanishes on this component, which implies $\alpha = 0$, so the pair (S, Q) belongs to the family described in Example 4.3(iii). Now suppose we are in the case of Example 4.3(ii), so $f_2 = 0$ and $f_3 = \alpha$. If σ sends the component of $\mathcal{C}_Q(5)$ given by $p = 0$ to a fiber of φ , then similarly $F_5 = q(3pq + \alpha)$ vanishes on this component, which implies $\alpha = 0$, so again the pair (S, Q) belongs to the family described in Example 4.3(iii). \square

Corollary 5.6. *Let k, S , and Q be as before. Assume that the pair (S, Q) is not isomorphic to the one of the pairs described in Examples 4.1, 4.2(iii), and 4.3(iii). If the order of Q in $S_0^{\text{ns}}(k)$ is 5, then also assume that the characteristic of k is not equal to 5. Then the rational map $\sigma : \mathcal{C}_Q(5) \rightarrow S$ sends at least one component of $\mathcal{C}_Q(5)$ to a horizontal curve on S .*

Proof. From the assumption $y_0 \neq 0$, it follows that the order of Q is at least 3. The statement now follows immediately from Propositions 5.2, 5.3, 5.4, and 5.5. \square

Remark 5.7. Generically, the surface $T \subset \mathbb{P}$, which is the closure of the union of all $C \in \mathcal{C}_Q(5)$, has degree 12, and the curve $\sigma(\overline{\mathcal{C}_Q(5)})$ is the intersection of S with T and therefore linearly equivalent

with $-12K_S$. The arithmetic genus of $\sigma(\overline{\mathcal{C}}_Q(5))$ is 67 in this case, and $\sigma(\overline{\mathcal{C}}_Q(5))$ has multiplicity 10 at the point Q , multiplicity 2 at $-5Q$, and is also singular at 20 more points, which agrees with the fact that the geometric genus equals

$$67 - \frac{1}{2} \cdot 10 \cdot (10 - 1) - \frac{1}{2} \cdot 2 \cdot (2 - 1) - 20(\frac{1}{2} \cdot 2 \cdot (2 - 1)) = 1.$$

These 22 singular points of $\sigma(\overline{\mathcal{C}}_Q(5))$ are the intersection points of S with the singular locus of T , which is a curve with an embedded point at Q . Moreover, the family of intersections of S with a hypersurface of degree 12 has dimension 78, as can be seen from the fact that the space of polynomials in x, y, z, w of weighted degree 12 modulo the multiples of the defining equation of S has dimension $102 - 23 = 79$ or from the fact that the linear system of curves in \mathbb{P}^2 of degree $3d$ having multiplicity at least d at each of 8 given points has dimension $\binom{3d+2}{2} - 1 - 8 \cdot \binom{d+1}{2} = \binom{d+1}{2}$, which equals 78 for $d = 12$. Hence, the subfamily of those intersections that have multiplicity 10 at Q , multiplicity 2 at $-5Q$, and 20 more singularities, has dimension

$$78 - \frac{1}{2} \cdot 10 \cdot (10 + 1) - \frac{1}{2} \cdot 2 \cdot (2 + 1) - 20 = 0,$$

so there are only finitely many curves satisfying these conditions.

Note that the projection $\nu: S \rightarrow \mathbb{P}(2, 1, 1)$ from S to the weighted projective space with coordinates x, z, w , gives S the structure of a double cover of a cone that is ramified at the singular point \mathcal{O} (corresponding to the vertex of the cone), as well as over the curve given by $x^3 + fx + g = 0$, i.e., the locus of nontrivial 2-torsion points. The involution induced by this double cover is multiplication by -1 on the elliptic fibration. If we let $-\sigma(\overline{\mathcal{C}}_Q(5)) \subset S$ denote the image of $\sigma(\overline{\mathcal{C}}_Q(5))$ under this involution, then $\sigma(\overline{\mathcal{C}}_Q(5))$ and $-\sigma(\overline{\mathcal{C}}_Q(5))$ intersect each other in 36 points on the ramification locus of ν , as well as 108 points off the ramification locus. The image $\nu(\sigma(\overline{\mathcal{C}}_Q(5))) = \nu(-\sigma(\overline{\mathcal{C}}_Q(5))) \subset \mathbb{P}(2, 1, 1)$ is a curve of degree 24 that intersects the branch locus of ν at 36 points, being tangent at each, that has multiplicity 10 at $\nu(Q) = [x_0 : 0 : 1]$, multiplicity 2 at $\nu(-5Q)$, and that is singular at 74 more points (namely the 20 images under ν of the remaining singular points of $\sigma(\overline{\mathcal{C}}_Q(5))$, and the 54 images of the intersection points of $\sigma(\overline{\mathcal{C}}_Q(5))$ and $-\sigma(\overline{\mathcal{C}}_Q(5))$). These properties narrow down the 168-dimensional family of curves in $\mathbb{P}(2, 1, 1)$ of degree 24 to only finitely many curves.

Remark 5.8. Given that for all pairs (S, Q) described in the examples in the previous section there are at least six (-1) -curves on S going through Q , whenever we want to exclude any of these examples, it suffices to assume that Q does not lie on six (-1) -curves. The fact that the existence of (-1) -curves on S through Q is related to the existence of a component of $\overline{\mathcal{C}}_Q(5)$ that maps under $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$ to a fiber of $\varphi: S \rightarrow \mathbb{P}^1$, can be understood from the intersection of S with the surface T (cf. Remark 5.7). The pull-back of $S \cap T$ under the blow-up $\mathcal{E} \rightarrow S$ is a divisor D on \mathcal{E} that consists of

- (i) the components of the strict transform D_0 of the image $\sigma(\overline{\mathcal{C}}_Q(5))$,
- (ii) the strict transforms of the (-1) -curves on S through Q ,
- (iii) the fiber \mathcal{E}_0 , and
- (iv) the zero section \mathcal{O}

with certain multiplicities. The degree of the restriction of $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ to D equals the intersection number of D with any fiber of π . Generically, this degree equals $\deg T = 12$, the multiplicities of \mathcal{E}_0 and \mathcal{O} are 0, and the multiplicities of the components in (i) and (ii) are 1; the (-1) -curves on S intersect fibers with multiplicity 1, so the restriction of π to D_0 equals $12 - s$, where s is the number of (-1) -curves on S through Q . In general, the multiplicities of the components in (iii) and (iv) seem to depend only on the order of Q in $\mathcal{E}_0^{\text{ns}}(k)$ and the singularity type of \mathcal{E}_0 , but in any case we find that the more (-1) -curves there are on S that go through Q , the smaller the degree of the restriction of π to D_0 , forcing all components of D_0 to be vertical in extreme cases. In fact, a thorough investigation of the degree of T as well as all multiplicities might yield another proof of Corollary 5.6 under the assumption that Q not lie on six (-1) -curves of S , but it is not clear that this will require less computational effort than the given proof, especially given that even in the generic case, the intersection $S \cap T$ does not appear to admit a very elegant description (cf. Remark 5.7).

6. TORSION IN A BASE CHANGE

In this section, k is still a field of characteristic not equal to 2 or 3.

Lemma 6.1. *Let B be a smooth curve over k and $\pi: \mathcal{E} \rightarrow B$ a minimal nonsingular elliptic fibration. Let C be a smooth curve over k and $\tau: C \rightarrow B$ a nonconstant morphism. Let $\pi': \mathcal{E}' \rightarrow C$ be the minimal nonsingular model of the base change $\mathcal{E} \times_B C \rightarrow C$ of π by τ . Let $c \in C$, set $b = \tau(c)$, and let T and T' be the types of the fibers of π and π' over b and c , respectively. Let $e = e_c(\tau)$ be the ramification index of τ at c . Then the following statements hold.*

- (1) *If $T = I_d$ for some integer d , then $T' = I_{de}$.*
- (2) *If $T = I_d^*$ for some integer d , then $T' = I_{de}$ for even e and $T' = I_{de}^*$ for odd e .*
- (3) *If $T = IV^*$, then $T' = I_0, IV^*, IV$ for $e \equiv 0, 1, 2 \pmod{3}$, respectively.*
- (4) *If $T = II$, then $T' = I_0, II, IV, I_0^*, IV^*, II^*$ for $e \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$, respectively.*
- (5) *If $T = III$, then $T' = I_0, III, I_0^*, III^*$ for $e \equiv 0, 1, 2, 3 \pmod{4}$, respectively.*

Proof. This follows directly from Tate's algorithm (see [33] and [32, IV.9.4]). See also [19, Table VI.4.1], which is stated for characteristic zero. \square

Lemma 6.2. *Suppose k is algebraically closed. Let S be a del Pezzo surface of degree 1 over k and $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ the associated elliptic fibration. Let M and N denote the number of singular fibers of π of type I_1 and type II , respectively. Then we have $M + 2N = 12$. If π is not isotrivial, then we have $M \geq 4$.*

Proof. In characteristic zero, the identity $M + 2N = 12$ follows from the more general fact that the Euler number of \mathcal{E} , which is 12, equals the sum of the local Euler numbers, which are 1 and 2 for fibers of type I_1 and II , respectively (see [19, Table IV.3.1] and [19, Lemma IV.3.3]). The inequality $M < 4$ implies $N \geq 5$, which implies $N = 6$ by [24, Lemma 1.2], which in turn implies that π is isotrivial. Since both [19] and [24] assume that the characteristic is zero, we provide an alternative direct argument for arbitrary characteristic (not equal to 2 or 3) here. Let $f, g \in k[z, w]$ be such that S is given by (1). The discriminant $\Delta = 4f^3 + 27g^2$ vanishes at points of \mathbb{P}^1 corresponding to nodal (I_1) and cuspidal (II) fibers to order 1 and 2, respectively, so we get $M + 2N = \deg \Delta = 12$. For any $t \in \mathbb{P}^1(\bar{k})$ for which $\pi^{-1}(t)$ has type II , both Δ and $j = 2^8 3^3 f^3 / \Delta$ vanish at t (see [32, Tate's Algorithm, IV.9.4]), which implies that f vanishes at t . It follows that f vanishes at at least N points, so if $M < 4$, i.e. $N \geq 5$, then $f = 0$, so π is isotrivial. \square

For $n \geq 3$, let $Y_1(n)$ be the usual modular curve over $\mathbb{Z}[1/n]$, parametrizing elliptic curves with a point of order n in the sense that there exists a universal elliptic curve $E(n) \rightarrow Y_1(n)$ with a section P that has order n in every fiber, such that every elliptic curve E over a scheme S over $\mathbb{Z}[1/n]$ —with nowhere vanishing j -invariant if $n = 3$ —with a section that has order n in every fiber, is the base change of $E(n)/Y_1(n)$ by a unique morphism $S \rightarrow Y_1(n)$. Let $X_1(n)$ be the usual projective closure of $Y_1(n)$, and let $v(n): \mathbb{E}(n) \rightarrow X_1(n)$ be the minimal nonsingular elliptic fibration over $X_1(n)$ associated to $E(n)/Y_1(n)$. From Ogg's description of the cusps of $X_1(n)$ in [23], we conclude that for each $n \geq 5$ and each divisor d of n , the number of fibers of $v(n)$ of type I_d is $\frac{1}{2}\varphi(d)\varphi(n/d)$, cf. [15, Table 3 and p. 219]. Table 1 describes the singular fibers of $v(n)$ for several n (see [29, Proposition 4.2]). To parametrize elliptic curves over a field of characteristic p

n	$g(X_1(n))$	sing. fibers of $v(n)$
3	0	$IV^* + I_3 + I_1$
5	0	$2I_5 + 2I_1$
7	0	$3I_7 + 3I_1$
11	1	$5I_{11} + 5I_1$

TABLE 1. Singular fibers of $v(n)$

with a point of order p , we use Igusa curves instead of the modular curves above. For an extensive treatise of the subject, we refer the reader to [9] and [13, Chapter 12]. For any prime $p \geq 3$, the smooth affine Igusa curve $\text{Ig}(p)^{\text{ord}}$ over \mathbb{F}_p parametrizes ordinary elliptic curves E with a point

that generates the kernel of the Verschiebung map in the following sense (see [13, Section 12.3 and Corollary 12.6.3]). For every scheme S over \mathbb{F}_p , the absolute Frobenius $S \rightarrow S$ is the map that corresponds on affine rings to the map $x \rightarrow x^p$. For every elliptic curve $E \rightarrow S$, we let $E^{(p)} \rightarrow S$ denote the base change of $E \rightarrow S$ by the absolute Frobenius $S \rightarrow S$. By the universal property of the fibered product, the absolute Frobenius $E \rightarrow E$ factors as the composition of the projection $E^{(p)} \rightarrow E$ and a map $F = F_{E/S}: E \rightarrow E^{(p)}$ that we call the *relative Frobenius*. The dual isogeny $V = V_{E/S}: E^{(p)} \rightarrow E$ of $F_{E/S}$ is called the *Verschiebung*. There exists an elliptic curve $\mathfrak{E}(p)^\circ$ over the Igusa curve $\text{Ig}(p)^{\text{ord}}$, as well a section \mathfrak{P} of the associated elliptic curve $\mathfrak{E}(p)^{\circ(p)} \rightarrow \text{Ig}(p)^{\text{ord}}$, such that all fibers of both fibrations are ordinary and \mathfrak{P} generates the kernel of the Verschiebung $V: \mathfrak{E}(p)^{\circ(p)} \rightarrow \mathfrak{E}(p)^\circ$, and such that for every elliptic curve E over a scheme S over \mathbb{F}_p of which all fibers are ordinary, with a section P of the associated curve $E^{(p)} \rightarrow S$ that generates the kernel of the Verschiebung $V: E^{(p)} \rightarrow E$, there is a unique morphism $\alpha: S \rightarrow \text{Ig}(p)^{\text{ord}}$ such that $E, E^{(p)}$, and P are the base change of $\mathfrak{E}^\circ, \mathfrak{E}(p)^{\circ(p)}$, and \mathfrak{P} , respectively, by α .

If k is a field of characteristic p and E' is an elliptic curve over $S = \text{Spec } k$ with a point P of order p , then there is an elliptic curve $E \rightarrow S$ such that $E^{(p)} \rightarrow S$ is isomorphic to $E' \rightarrow S$ and P generates the kernel of Verschiebung; hence $E' \rightarrow S$ is a base change of the universal curve $\mathfrak{E}(p)^{\circ(p)} \rightarrow \text{Ig}(p)^{\text{ord}}$.

Let $\overline{\text{Ig}(p)^{\text{ord}}}$ denote the nonsingular projective completion of $\text{Ig}(p)^{\text{ord}}$, and let $\omega(p): \mathfrak{E}(p)^{(p)} \rightarrow \overline{\text{Ig}(p)^{\text{ord}}}$ denote the minimal nonsingular projective model of $\mathfrak{E}(p)^{\circ(p)} \rightarrow \text{Ig}(p)^{\text{ord}}$. In the proof of Theorem 6.3 we will use the fiber types of the singular fibers of $\omega(p)$, which are given in Table 2. The fibers at the $(p-1)/2$ cusps have type I_p [17, Theorem 10.3] and the type of the fibers above the supersingular points can be deduced from [17, Theorem 10.1]; for $p = 13$ it suffices to note that the only supersingular j -value modulo 13 is 5, while for $p \in \{5, 7, 11\}$, the fibers are also given in [11, Proposition 1.3].

p	$g(\overline{\text{Ig}(p)^{\text{ord}}})$	sing. fibers of $\omega(p)$
5	0	$2I_5 + II$
7	0	$3I_7 + III$
11	0	$5I_{11} + II + III$
13	1	$6I_{13} + I_0^*$

TABLE 2. Singular fibers of $\omega(p)$

Theorem 6.3. *Let S be a del Pezzo surface of degree 1 over k and $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$ the associated elliptic fibration. Let C be a smooth, connected curve over k of genus at most 1, and $\tau: C \rightarrow \mathbb{P}^1$ a nonconstant morphism. Then the base change $\mathcal{E} \times_{\mathbb{P}^1} C \rightarrow C$ of π by τ has no nonzero section of finite order.*

Proof. Without loss of generality, we assume that C is projective and that k is algebraically closed. As the curve C is smooth and connected, it is integral, so it has a unique generic point η that is dense in C . The curve $\mathcal{E} \times_{\mathbb{P}^1} \eta$ is an elliptic curve over the function field $\kappa(C)$ of C , which is an extension of the function field $k(t)$ of \mathbb{P}^1 with $t = z/w$. Let $j \in k(t)$ be the j -invariant of the generic fiber of π . Assume the elliptic fibration $\mathcal{E} \times_{\mathbb{P}^1} C \rightarrow C$ has a nonzero section of finite order, say order $n > 1$. Then the curve $\mathcal{E} \times_{\mathbb{P}^1} \eta$ has a point of order n over $\kappa(C)$. Without loss of generality, we assume n is prime. Let $f, g \in k[z, w]$ be homogeneous polynomials such that S is isomorphic to the surface in \mathbb{P} given by (1). Let M and N denote the number of fibers of π of type I_1 (nodal) and II (cuspidal), respectively. Then $M + 2N = 12$ by Lemma 6.2. We will show that the genus of C is at least 2 by considering several cases, thus deriving the contradiction that proves the statement.

I) We first consider the case $n = 2$. Note that π itself has no section of order 2, for if it did, it would be given by $y = 0$ and $x = h(z, w)$ for some homogeneous polynomial $h \in k[z, w]$ of degree 2, and then S would be singular at the point on this section in the fibers given by $3h^2 + f = 0$. Since the locus $L \subset S$ of the 2-torsion points has degree 3 over $\mathbb{P}^1(z, w)$, it follows that L is irreducible.

To compute its genus, note that the map $\lambda: L \rightarrow \mathbb{P}^1$ ramifies whenever $D = 4f^3 + 27g^2$ vanishes. Moreover, if D vanishes to order 1 at $t \in \mathbb{P}^1$, which happens if and only if the fiber \mathcal{E}_t of π has type I_1 , then there are two points on L above t with ramification indices 1 and 2, while if D vanishes to order 2, which happens if and only if the fiber \mathcal{E}_t has type II , then there is a unique point on L above t with ramification index 3. It follows that the degree of the ramification divisor of λ equals $M + 2N = 12$, so the Riemann-Hurwitz formula applied to λ shows that the genus of L equals $1 + \frac{1}{2}(-2(\deg \lambda) + 12) = 4$. If the base change of π by τ has a section of order 2, then this section would map nontrivially to L , so we get $g(C) \geq 4$.

II) We now consider the case that j is constant, that is $j \in k$, and may assume $n \neq 2$. Then there are $a, b \in k$ and $h \in \overline{k(t)}$, where $\overline{k(t)}$ denotes an algebraic closure of $k(t)$, such that $f(t, 1) = ah^2$ and $g(t, 1) = bh^3$. If $f, g \neq 0$, then $h = ab^{-1}g(t, 1)f(t, 1)^{-1}$ is contained in $k(t)$ and one checks that S is not smooth. If $g = 0$, then S is not smooth either, so we find $f = 0$ and again from smoothness of S , we find that $g(t, 1)$ is separable and has degree 5 or 6 in t (cf. [38, Proposition 3.1]). Suppose $P = (x_1, y_1) \in \mathcal{E} \times_{\mathbb{P}^1} \eta$ is a point over $\kappa(C)$ of order n . Let $\overline{\kappa(C)}$ be an algebraic closure of $\kappa(C)$ and $\beta \in \overline{\kappa(C)}$ an element satisfying $\beta^6 = g(t, 1)$. Then $(x_1\beta^{-2}, y_1\beta^{-3})$ is a point of order n on the curve given by $y^2 = x^3 + 1$, so there are $x_0, y_0 \in k$ such that $x_1 = x_0\beta^2$ and $y_1 = y_0\beta^3$. From $n \neq 2$, we get $y_1 \neq 0$. If $x_1 \neq 0$, then $\beta = x_0y_0^{-1}y_1x_1^{-1}$ is contained in $\kappa(C)$; if $x_1 = 0$, then $y_1^2 = g(t, 1)$, so in any case, $g(t, 1)$ is a square in $\kappa(C)$, which implies that $\kappa(C)$ contains a subfield of genus 2, so C has at least genus 2 itself.

III) The case $n \geq 3$ and $j \notin k$. If the characteristic of k is not equal to n , then we set $Y = Y_1(n)$ and $X = X_1(n)$ and $\mathbb{E} = \mathbb{E}(n)$ and $v = v(n)$; otherwise, we set $Y = \text{Ig}(n)^{\text{ord}}$ and $X = \overline{\text{Ig}(n)}^{\text{ord}}$ and $\mathbb{E} = \mathfrak{E}(n)^{(n)}$ and $v = \omega(n)$. In either case, there is a morphism $\eta \rightarrow Y \subset X$ such that the elliptic curve $\mathcal{E} \times_{\mathbb{P}^1} \eta$ over η is the base change of \mathbb{E} over X . This morphism extends to a morphism $\chi: C \rightarrow X$, which is nonconstant because j is not constant. The elliptic surfaces $\mathcal{E} \times_{\mathbb{P}^1} C$ and $\mathbb{E} \times_X C$ have isomorphic generic fibers $\mathcal{E} \times_{\mathbb{P}^1} \eta \cong \mathbb{E} \times_X \eta$, so their minimal nonsingular models are isomorphic as well by [19, Proposition II.1.2 and Corollary II.1.3]. Let $\pi': \mathcal{E}' \rightarrow C$ be this minimal nonsingular elliptic fibration.

$$\begin{array}{ccccccc}
 \mathcal{E} & \longleftarrow & \mathcal{E} \times_{\mathbb{P}^1} C & \xleftarrow{\quad} & \mathcal{E}' & \xrightarrow{\quad} & \mathbb{E} \times_X C \longrightarrow \mathbb{E} \\
 \downarrow \pi & & \downarrow & \nearrow \pi' & \searrow \pi' & & \downarrow v \\
 \mathbb{P}^1 & \xleftarrow{\quad \tau \quad} & C & \xrightarrow{\quad \chi \quad} & C & \xrightarrow{\quad \chi \quad} & X
 \end{array}$$

Set $d = \deg \tau$ and let $R \in \text{Div } C$ denote the ramification divisor of τ . Then the degree of R is at least

$$(15) \quad \sum_{c \in C} (e_c(\tau) - 1) \geq \sum_{\substack{b \in \mathbb{P}^1 \\ \mathcal{E}_b \text{ type } I_1}} \left(\sum_{\substack{c \in C \\ \tau(c)=b}} (e_c(\tau) - 1) \right) + \sum_{\substack{b \in \mathbb{P}^1 \\ \mathcal{E}_b \text{ type } II}} \left(\sum_{\substack{c \in C \\ \tau(c)=b}} (e_c(\tau) - 1) \right),$$

where $e_c(\tau)$ denotes the ramification index of τ at c .

Lemma 6.1 implies that the points $c \in C$ for which the fiber $\mathcal{E}_{\tau(c)}$ of π above $\tau(c)$ has type I_1 are exactly the points for which the fiber \mathcal{E}'_c of π' above c has type I_m for some integer $m \geq 1$, and exactly the points for which the fiber $\mathbb{E}_{\chi(c)}$ of v above $\chi(c)$ has type I_j or I_j^* for some integer $j \geq 1$; for such c, m , and j , the quotient $\ell = m/j$ is a positive integer and we have $e_c(\tau) = j\ell$ and $e_c(\chi) = \ell$. For each $j \geq 1$, let r_j denote the number of fibers of v of type I_j or I_j^* ; for each $\ell \geq 1$, let $s_{j,\ell}$ denote the number of fibers of π' of type $I_{j\ell}$ that lie above a point $c \in C$ for which the fiber of v above $\chi(c)$ has type I_j or I_j^* . For every $x \in X$ we have $\sum_{c \in \chi^{-1}(x)} e_c(\chi) = \deg \chi$. Summing over all $x \in X$ for which the fiber \mathbb{E}_x has type I_j , we find $\sum_{\ell \geq 1} \ell s_{j,\ell} = (\deg \chi) \cdot r_j$ for all $j \geq 1$. The same argument applied to τ yields

$$Md = \sum_{j, \ell \geq 1} j \ell s_{j,\ell} = (\deg \chi) \cdot \sum_{j \geq 1} j r_j.$$

It follows that the first term of the right-hand side of (15) equals

$$(16) \quad \sum_{j, \ell \geq 1} (j\ell - 1)s_{j, \ell} \geq \sum_{j, \ell \geq 1} (j - 1)\ell s_{j, \ell} = (\deg \chi) \cdot \sum_{j \geq 1} (j - 1)r_j = \frac{\sum_{j \geq 1} (j - 1)r_j}{\sum_{j \geq 1} jr_j} \cdot Md.$$

We consider two subcases.

A) The characteristic of k is not equal to n . From $g(X) \leq g(C) \leq 1$ we conclude $n \leq 12$ or $n = 14$ or $n = 15$ (see [22, p. 109]), and since $n \geq 3$ is prime, we have $n \in \{3, 5, 7, 11\}$. From Table 1 above, we find that the fraction in the right-most expression of (16) is at least $\frac{1}{2}$. Since $v = v(n)$ has only fibers of type IV^* , I_1 , and I_n , Lemma 6.1 implies that π' does not have fibers of type II , II^* , or I_0^* . Again from Lemma 6.1, this time viewing π' as the minimal model of the base change of π by τ , we find that for every $c \in C$ for which the fiber of π above $\tau(c)$ has type II , the ramification index $e_c(\tau)$ is even, so we have $e_c(\tau) - 1 \geq \frac{1}{2}e_c(\tau)$. Therefore, the second term of the right-hand side of (15) is at least $\frac{1}{2}Nd$, so the degree of R is at least $\frac{1}{2}Md + \frac{1}{2}Nd \geq \frac{1}{4}d(M + 2N) = 3d$. The Riemann-Hurwitz formula applied to τ then yields $2g - 2 = -2d + \deg R \geq d > 0$, so $g > 1$.

B) The characteristic of k is equal to n . From $g(X) \leq g(C) \leq 1$ we conclude $n \in \{5, 7, 11, 13\}$ (for a formula for the genus of X , see [9, p. 96 and 99]). From Table 1 above, we find that the fraction in the right-most expression of (16) is at least $\frac{4}{5}$. Also, from the fact that π is not isotrivial, we get $M \geq 4$ by Lemma 6.2, so the degree of R is at least $\frac{4}{5}Md > 3d$. As before, the Riemann-Hurwitz formula yields $g > 1$. \square

7. PROOF OF THE MAIN THEOREMS

In this section, the field k is still of characteristic different from 2 and 3.

Theorem 7.1. *Suppose k is infinite. Let $S \subset \mathbb{P}$ be a del Pezzo surface of degree 1 over k , given by (1) for some homogeneous $f, g \in k[z, w]$ of degree 4 and 6, respectively. Let $Q = [x_0 : y_0 : 0 : 1] \in S(k)$ be a rational point with $y_0 \neq 0$. Suppose that the following statements hold.*

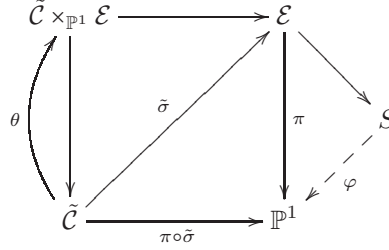
- *If the order of Q in $S_0^{\text{ns}}(k)$ is at least 4, then $\mathcal{C}_Q(5)$ has infinitely many k -points.*
- *If the order of Q in $S_0^{\text{ns}}(k)$ is 5, then the characteristic of k is not equal to 5.*
- *The pair (S, Q) is not isomorphic to a pair described in Examples 4.1, 4.2(iii), or 4.3(iii).*
- *If the pair (S, Q) is isomorphic to a pair described in Subexamples 4.2(i) or 4.3(i), then the set of k -points on $\mathcal{C}_Q(5)$ is Zariski dense in $\mathcal{C}_Q(5)$.*

Then the set $S(k)$ of k -points on S is Zariski dense in S .

Proof. Given S and Q , we let the curve $\mathcal{C}_Q(5)$, its completion $\overline{\mathcal{C}}_Q(5)$, the rational map $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$, the elliptic fibration $\pi: \mathcal{E} \rightarrow \mathbb{P}^1$, and the element $\phi_3 \in k$ be as in Sections 2 and 3. We claim that there exists an irreducible component \mathcal{C} of $\overline{\mathcal{C}}_Q(5)$ for which $\sigma(\mathcal{C})$ is a horizontal curve on S and $\mathcal{C}(k)$ is infinite. Indeed, if the order of Q in $S_0^{\text{ns}}(k)$ is at least 4, then $\phi_3 \neq 0$, so $\mathcal{C}_Q(5)$ is a double cover of $\mathbb{A}^1(p)$, the curve $\mathcal{C}_Q(5)$ has at most two irreducible components, and if there are two, then there is an involution that switches them, so the first assumption of the theorem implies that $\mathcal{C}_Q(5)(k)$ is Zariski dense in $\mathcal{C}_Q(5)$; thus, there exists an irreducible component \mathcal{C} of $\mathcal{C}_Q(5)$ that satisfies the claim by Corollary 5.6. If, on the other hand, the order of Q is 3, then for any pair (S, Q) that is not isomorphic to one of the pairs described in Examples 4.2 and 4.3, the unique component of $\mathcal{C}_Q(5)$ that projects birationally to $\mathbb{A}^1(p)$ satisfies the claim by Proposition 5.5, while for any pair (S, Q) that is isomorphic to one of the pairs described in those examples, any other component \mathcal{C} of $\mathcal{C}_Q(5)$ satisfies the claim, as its image is horizontal by Proposition 5.5 and density of $\mathcal{C}(k)$ follows either automatically in the case of Subexample (ii) or by assumption in the case of Subexample (i).

Let \mathcal{C} be a component of $\overline{\mathcal{C}}_Q(5)$ as in the claim, and let $\tilde{\mathcal{C}}$ be a normalization of \mathcal{C} . Then the rational map $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$ induces a morphism $\tilde{\sigma}: \tilde{\mathcal{C}} \rightarrow \mathcal{E}$. The composition $\pi \circ \tilde{\sigma}: \tilde{\mathcal{C}} \rightarrow \mathbb{P}^1$ corresponds on an open subset to the rational map $\varphi \circ \sigma: \overline{\mathcal{C}}_Q(5) \rightarrow \mathbb{P}^1$, so it is surjective by the

claim. Let θ denote the section $\text{id} \times \tilde{\sigma}: \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}} \times_{\mathbb{P}^1} \mathcal{E}$ of the elliptic surface $\tilde{\mathcal{C}} \times_{\mathbb{P}^1} \mathcal{E} \rightarrow \tilde{\mathcal{C}}$.



The section θ is not the zero section because $\sigma: \overline{\mathcal{C}}_Q(5) \rightarrow S$ sends curves $C \in \mathcal{C}_Q(5)$ that are not contained in S to a point unequal to $\mathcal{O} \in S$. The genus of $\tilde{\mathcal{C}}$ is at most 1, so by Theorem 6.3, the section θ has infinite order. Since $\tilde{\mathcal{C}}(k)$ is Zariski dense in $\tilde{\mathcal{C}}$, it follows that the rational points are dense on the images of all infinitely many multiples of θ . Thus, the k -rational points are dense in the surface $\tilde{\mathcal{C}} \times_{\mathbb{P}^1} \mathcal{E}$ and as this surface maps dominantly to S , we conclude that $S(k)$ is Zariski dense in S . \square

Obviously, for any point $Q' \in S(k) - \{\mathcal{O}\}$, we may apply an automorphism of $\mathbb{P}^1(z, w)$ to ensure that we have $\varphi(Q') = [0 : 1]$, so the implicit assumption in Theorem 7.1 and related statements that $\varphi(Q) = [0 : 1]$ is not a restriction.

Note that if the order of Q in $S_0^{\text{ns}}(k)$ is 3 and the pair (S, Q) is not isomorphic to a pair described in Examples 4.2(i) and 4.3(i), then the hypotheses of Theorem 7.1 are automatically satisfied, without further assumptions on $\mathcal{C}_Q(5)$.

Example 7.2. We took a small sample of approximately a hundred randomly chosen del Pezzo surfaces over \mathbb{Q} given by (1) with f and g having only coefficients 0, 1, and -1 . For nearly half of the cases, a short point search revealed a rational point Q for which we could show that it satisfies all conditions of Theorem 7.1, thus proving the rational points are Zariski dense. For the remaining cases, we could still find points Q , but the coefficients of $\mathcal{C}_Q(5)$ were too large to show that $\mathcal{C}_Q(5)$ has infinitely many rational points.

Example 7.3. As mentioned in Section 1, A. Várilly-Alvarado proves in [38, Theorem 2.1] that if we have $k = \mathbb{Q}$ and $f = 0$, and some technical conditions on g , as well as a finiteness conjecture hold, then the set of rational points is Zariski dense on the surface given by (1). He also mentions the surface S with $f = 0$ and $g = 243z^6 + 16w^6$ as an example that would not succumb to his methods, so we took S as a test example for our method. Unfortunately, the point $[0 : 4 : 0 : 1]$ of order 3 on $S_0 \subset S$ lies on nine (-1) -curves (cf. Example 4.3(iii)). It is not hard to find more rational points on this surface, but we did not succeed in finding any points on the curve $\mathcal{C}_Q(5)$ associated to any of these points Q as the coefficients are rather large: for the second-smallest point $Q = [-63 : 14 : 1 : 5]$, the conductor of the Jacobian of $\mathcal{C}_Q(5)$ has 62 digits. N. Elkies did prove that the points on S are dense with a different method [6].

Proof of Theorem 1.2. Without loss of generality, we assume $\varphi(Q) = [0 : 1]$, say $Q = [x_0 : y_0 : 0 : 1]$. The fact that Q does not have order 2 in $\mathcal{E}_0^{\text{ns}}(k)$ implies $y_0 \neq 0$, so that we may apply Theorem 7.1. The last hypothesis of Theorem 1.2 implies the last two of Theorem 7.1, which shows that $S(k)$ is indeed Zariski dense in S . \square

Proof of Theorem 1.3. Note that any point (x_0, y_0) on an elliptic curve given by $y^2 = x^3 + ax + b$ has order 3 if and only if $(a + 3x_0^2)^2 = 12x_0y_0^2$. Define the polynomials $f = \sum_{i=0}^4 f_i u^i$ and $g = \sum_{j=0}^6 g_j u^j$. Suppose we have $\ell \in \{0, \dots, 4\}$, $m \in \{0, \dots, 6\}$, and $\varepsilon > 0$. Since every elliptic curve over the real numbers \mathbb{R} has a nontrivial 3-torsion point, we may choose a nonzero rational number $t \in \mathbb{Q}^*$ and a point $Q = [x_0 : y_0 : t : 1] \in S(\mathbb{R})$ such that the fiber S_t given by $y^2 = x^3 + f(t)x + g(t)$ is smooth, the point Q has order 3 in $S_t(\mathbb{R})$, and Q does not lie on six (-1) -curves on S . Set $\xi_0 = \frac{1}{6}y_0^{-1}(f(t) + 3x_0^2)$, so that Q being 3-torsion implies $3\xi_0^2 = x_0$. Choose $\xi_1, y_1 \in \mathbb{Q}^*$ close to ξ_0

and y_0 , respectively, and set $x_1 = 3\xi_1^2$ and $Q' = [x_1 : y_1 : t : 1]$. Also set

$$\begin{aligned}\lambda &= f_\ell + t^{-\ell}(6\xi_1 y_1 - 3x_1^2 - f(t)), \\ \mu &= g_m + t^{-m}(y_1^2 - x_1^3 - (6\xi_1 y_1 - 3x_1^2)x_1 - g(t)), \\ f' &= f - f_\ell u^\ell + \lambda u^\ell, \\ g' &= g - g_m u^m + \mu u^m,\end{aligned}$$

so that f' and g' are the polynomials obtained from f and g after replacing f_ℓ and g_m by λ and μ , respectively. Then we have $f'(t) = 6\xi_1 y_1 - 3x_1^2$ and $g'(t) = y_1^2 - x_1^3 - f'(t)x_1$, so Q lies on the surface S' given by (2) with the two values f_ℓ and g_m replaced by λ and μ , respectively. If we choose ξ_1 and y_1 arbitrarily close to ξ_0 and y_0 , then λ and μ will be arbitrarily close to f_ℓ and g_m . By choosing them close enough, we also guarantee that S' and S'_t are smooth, and that Q' does not lie on six (-1) -curves on S' . From the identity $(f'(t) + 3x_1^2)^2 = 36\xi_1^2 y_1^2 = 12x_1 y_1^2$ we conclude that Q' has order 3 in $S'_t(\mathbb{Q})$, so we may apply Theorem 1.2, which yields that $S'(\mathbb{Q})$ is Zariski dense in S' . \square

Lemma 7.4. *Let k be an infinite field and $X \rightarrow \mathbb{P}^1$ an elliptic fibration over k with a nontorsion section. Then there are infinitely many points $t \in \mathbb{P}^1(k)$ for which the fiber X_t contains infinitely many k -rational points.*

Proof. If k is algebraic over a finite field, then this follows from the Weil conjectures. Otherwise, we replace k without loss of generality by an infinite subfield that is finitely generated over its prime subfield, over which everything is defined. Then k is either a number field or a transcendental extension of its prime field and in all cases, k is Hilbertian (see [8] for number fields and [7, Theorem 13.4.2] for a modern treatment of the general case). The lemma now follows immediately from Néron's Specialization Theorem [21, Théorème IV.6] (see also [16, Theorem 7.2] and [25, Remark 3.7(1)]). \square

Proof of Theorem 1.4. Without loss of generality, we assume that the nodal fiber lies above $[0 : 1]$. When Q runs over the nodal curve S_0 , the curves $\overline{\mathcal{C}}_Q(5)$ form a family of genus-one curves. More precisely, the equation in (10) describes a surface $X \subset \mathbb{A}^1(x_0) \times \mathbb{P}(1, 2, 1)$, and if $Q = [x_0 : y_0 : 0 : 1]$ is a point on S_0 with $y_0 \neq 0$ and not of order 3 in $S_0^{\text{ns}}(k)$, then the fiber of the projection $\mu: X \rightarrow \mathbb{A}^1$ above x_0 is isomorphic to $\overline{\mathcal{C}}_Q(5)$. The fibered product $(S_0 - \{\mathcal{O}\}) \times_{\mathbb{A}^1} X$ is the family of curves $\overline{\mathcal{C}}_Q(5)$, at least outside finitely many points $Q \in S_0$. Let $d \in k^*$ be such that $f_0 = -3d^2$ and $g_0 = 2d^3$. Note that we have two rational maps $\chi_i: \mathbb{A}^1 \rightarrow X$, for $i = 1, 2$, that are rational sections of μ , namely given by $x_0 \mapsto (x_0, [1 : \alpha_i : 0])$ with $\alpha_1 = \frac{1}{4}(x_0 + 2d)^{-1}$ and $\alpha_2 = \frac{1}{4}(x_0 + 7d)(x_0 + 2d)^{-1}(x_0 + 3d)^{-1}$ as in Lemma 3.4. These maps extend to morphisms and we choose χ_1 to be the zero section, making μ a Jacobian elliptic fibration.

We claim that the section χ_2 has infinite order. The model $X \rightarrow \mathbb{A}^1$ is highly singular, so instead we consider the surface $X' \subset \mathbb{A}^1(x_0) \times \mathbb{P}(1, 2, 1)$ that is the image of X under the birational map

$$\begin{aligned}\mathbb{A}^1 \times \mathbb{P}(1, 2, 1) &\rightarrow \mathbb{A}^1 \times \mathbb{P}(1, 2, 1) \\ (x_0, [\overline{p} : \overline{q} : \overline{r}]) &\mapsto (x_0, [\overline{p}' : \overline{q}' : \overline{r}'])\end{aligned}$$

with

$$\begin{aligned}\overline{p}' &= 8(x_0 - d)^2 \overline{p} + (x_0 - d)(f_1 d + g_1) \overline{r}, \\ \overline{q}' &= 2\varphi_2^{-1}(2c_1 \overline{q} + c_2 \overline{p}^2 + c_3 \overline{p} \overline{r} + c_4 \overline{r}^2), \\ \overline{r}' &= 8\overline{r},\end{aligned}$$

where $\phi_2, c_1, c_2, c_3, c_4$ depend on the now variable x_0 as they did before. Note that $f_1 d + g_1$ is nonzero because S is smooth. For $x_0 \notin \{d, -2d, -3d\}$, the fibers of $\mu: X \rightarrow \mathbb{A}^1$ and $\mu': X' \rightarrow \mathbb{A}^1$ are isomorphic. The model X' is given by $\overline{q}^2 = H(\overline{p}, \overline{r})$, where $H \in k[x_0][\overline{p}, \overline{r}]$ is homogeneous of degree 4. The fiber X'_d of μ' above $x_0 = d$ is given by

$$(17) \quad \overline{q}^2 = 81d^4(f_1 d + g_1) \overline{p}^2 \overline{r} (\overline{p} + (f_1 d + g_1) \overline{r}).$$

This fiber X'_d is singular at the point $(d, [0 : 0 : 1])$, and in fact so is X' , but the fiber is smooth everywhere else. The sections χ_1 and χ_2 correspond to the sections $\chi'_1: x_0 \mapsto (x_0, [4 : 6d(d-x_0) : 0])$ and $\chi'_2: x_0 \mapsto (x_0, [4 : 6d(x_0-d) : 0])$ of μ' , respectively. These sections intersect in the point $(d, [1 : 0 : 0])$, which is smooth in its fiber. Therefore, in a minimal nonsingular projective model $\bar{\mu}: \bar{X} \rightarrow \mathbb{P}^1$ of the fibration μ , the two sections intersect as well. Hence, χ'_2 is in the kernel of the reduction $\bar{X}(\mathbb{P}^1) \rightarrow \bar{X}_d(k)$, where \bar{X}_d is the fiber of $\bar{\mu}$ above $x_0 = d$. This kernel is isomorphic to a subgroup of the formal group associated to $\bar{\mu}$ (or μ') and the completion of $k[x_0]$ at the maximal ideal $(x_0 - d)$, cf. [31, Proposition VII.2.2]. By [31, Proposition IV.3.2(b)], all torsion elements of the formal group have p -power order, where p is the characteristic of k . This proves the claim for $p = 0$ (cf. [20, Theorem 1.1(a)]). We now assume $p > 0$ and determine the Kodaira type of the singular fiber \bar{X}_d of $\bar{\mu}$. One checks that the discriminant of H equals

$$\Delta = (x_0 - d)^3(x_0 + 2d)^8(x_0 + 3d)^2D(x_0),$$

where D is a polynomial of degree 35 satisfying $2^{11}D(d) = -3^{13}d^{11}(f_1d + g_1)^{12} \neq 0$. Hence, the valuation of Δ at $x_0 = d$ equals 3; the fiber X'_d described in (17) is nodal, so the reduction is multiplicative and we conclude from [32, Tate's Algorithm IV.9.4] that \bar{X}_d has type I_3 . It follows that the j -invariant of μ is not constant. Suppose that χ'_2 is torsion. Then $\bar{\mu}$ admits a section of order p , so there is a surjective morphism $\psi: \mathbb{P}^1 \rightarrow \overline{\text{Ig}(p)}^{\text{ord}}$ such that the generic fiber of $\bar{\mu}$ is isomorphic to the generic fiber of the base change of the fibration $\omega(p): \mathfrak{E}(p)^{(p)} \rightarrow \overline{\text{Ig}(p)}^{\text{ord}}$ by ψ (cf. part III of the proof of Theorem 6.3). This implies that the minimal nonsingular model of this base change is isomorphic to $\bar{\mu}$. However, the existence of ψ implies that $\overline{\text{Ig}(p)}^{\text{ord}}$ has genus 0, so $p \leq 11$, and from Lemma 6.1 and Table 2 we find that no base change of $\omega(p)$ has a minimal nonsingular model with fibers of type I_3 . This contradicts the fact that \bar{X}_d has type I_3 and the claim follows.

It follows that the section $(S_0 - \{\mathcal{O}\}) \rightarrow (S_0 - \{\mathcal{O}\}) \times_{\mathbb{A}^1} X$ induced by χ_2 also has infinite order on the elliptic fibration $(S_0 - \{\mathcal{O}\}) \times_{\mathbb{A}^1} X \rightarrow S_0 - \{\mathcal{O}\}$ with the section induced by χ_1 as zero section. After replacing S_0 by its normalization we may apply Lemma 7.4, which implies that the curve $\bar{\mathcal{C}}_Q(5)$ has infinitely many rational points for infinitely many $Q \in S_0^{\text{ns}}(k)$, in particular for some Q of order larger than 5 in $S_0^{\text{ns}}(k)$. Theorem 1.2 then shows that $S(k)$ is Zariski dense in S . \square

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